# MATHEMATICS MAGAZINE

Supplement to Volume 35, Number 1

Index to Volume 34 September 1960 through December 1961

## ARTICLES BY TITLE

## *ARTICLES BY AUTHORS*

## *PROBLEMS AND QUESTIONS*

### *Contributors*

Numbers refer to pages. Bold type indicates a problem solved and solution published; italics, a problem solved but solution not published; ordinary type, a problem proposed.

*Solutions*

The page on which a solution appears is in the parentheses following the number of the problem.



*Quickies*



*Trickies*



*Falsies*

## *BOOK REVIEWS*

Following the title of the book reviewed is the name of the reviewer and the page on which the review appeared.

# MATHEMATICS MAGAZINE

## CONTENTS




VOLUME 35 • JANUARY 1962 • NUMBER 1

MM

# MATHEMATICS MAGAZINE

ROBERT E. HORTON, Editor



---

EDITORIAL CORRESPONDENCE should be addressed to the Editor. Robert E. Horton, Department of Mathematics, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

---

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Los Angeles, California and Buffalo, New York, bi-monthly except July-August. Ordinary subscriptions are: 1 year $3.00; 2 years $5.75; 3 years $8.50: 4 years $11.00; 5 years $13.00. Members of the Mathematical Association of America may subscribe at the special rate of 2 years for $5.00. Single copies are 65c; reprints, bound 2c per page plus 10c each, provided the order is placed before the article goes to press.

---

*Published with the assistance of the Jacob Houck Memorial Fund of the Mathematical Association of America.*

---

Second class postage paid in Los Angeles, California.

---

# THE EDITOR'S PAGE

*Undergraduate Research*

Most of the collegiate world is busy creating better academic programs for their more gifted students. One outcome of this activity at many colleges is some sort of plan for stimulating undergraduate research. A conference on undergraduate research in mathematics was held in June, 1961 at Carleton College, Northfield, Minnesota. At this conference a need was expressed for some avenues for the publication of worth-while undergraduate research.

From time to time for the past fifteen years the *Mathematics Magazine* has published papers written by college students and even high school students. We encourage this. Our only requirement is that such papers must pass the judgment of the editors on the same basis as papers received from any other source.

In this issue we publish a paper by a student, Mr. T. A. Chapman, on the Kuratowski closure theorem. This paper was forwarded by Professor H. W. Gould of West Virginia University where Mr. Chapman is an undergraduate. Professors everywhere are invited to encourage their gifted undergraduates to submit their original work for consideration. Acceptance of these papers will depend upon their mathematical value, not upon who wrote them.

# FIBONACCI SEQUENCES AND A GEOMETRICAL PARADOX

A. F. HORADAM, University of New England, Armidale, N. S. W., Australia

**1. The Paradox.** A well-known geometrical paradox, which Rouse Ball [1] traces to the ZEITSCHRIFT FÜR MATHEMATIK UND PHYSIK, Leipzig, 1868, Vol. XIII, p. 162, requires us to subdivide a square of side

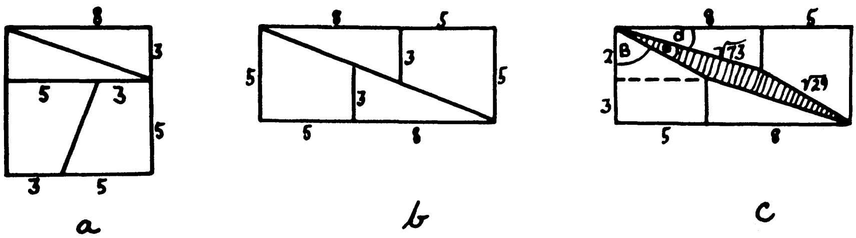


Fig. 1

8 units as shown in Fig. 1 (a) and rearrange it to form a rectangle of sides 5 and 13 units as shown in Fig. 1 (b). The area of the square is $8 \times 8 = 64$ square units, while that of the rectangle is $5 \times 13 = 65$ square units. Seemingly, we have gained an area of one square unit. But how? The answer, also well-known, is that if the subdivision is done accurately, there will appear within the rectangle a small parallelogram of unit area, with sides $\sqrt{29}$, $\sqrt{73}$ units, as shown in Fig. 1 (c). It is a matter of elementary trigonometry to calculate that $\theta \doteqdot 1°15'$.

The relationship $5 \times 13 - 8^2 = 1$ is a particular example of a result connecting three successive Fibonacci numbers $F_n$, $F_{n+1}$, $F_{n+2}$ in the Fibonacci sequence.

Here, I wish (i) to extend the paradox to cover all sets of three consecutive Fibonacci numbers; (ii) to establish a formula for $\theta$; and (iii) to generalize the paradox by means of a generalized Fibonacci sequence.

Therefore, we need to know a little about the Fibonacci sequence.

**2. Fibonacci's sequence.** The *Fibonacci sequence*

$$(1) \qquad f: \begin{matrix} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & \cdots \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & \cdots \end{matrix}$$

defined by

$$(2) \qquad F_1 = F_2 = 1$$

$$(3) \qquad F_{n+2} = F_{n+1} + F_n \quad \text{(recurrence relation)}$$

has, as *nth Fibonacci number,*

$$(4) \qquad F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$$

where

(5) $$a = \frac{1+\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2} \quad \text{(i. e., } -1 < b < 0)$$

so that

(6) $$a+b = 1, \quad ab = -1 \ \text{(i. e., } b = -\tfrac{1}{a}), \quad a-b = \sqrt{5}, \quad \lim_{n\to\infty} b^n = 0 .$$

Using (4), (5) and (6) we may establish the formulae

(7) $$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$$

(8) $$F_n^2 + F_{n-1}^2 = F_{2n-1},$$

due originally to Simson and Catalan. Also, from (4) and (6), we have

(9) $$\lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}}\right) = \frac{1}{a},$$

where $(1/a) = .618\cdots$ is the well-known "golden section" ratio discussed by Euclid, but probably of greater antiquity (perhaps of Pythagorean origin).

**3. General solution of the paradox.** From (7), we see that we must consider the two cases: $F_n F_{n+2} - F_{n+1}^2 = \pm 1$. In the standard case of the paradox (section 1), the relevant numbers are 5, 8, 13, i. e., $F_5$, $F_6$, $F_7$, i. e., $n$ (= 5) is odd. Geometrically, $n$ odd means that there is a unit parallelogram as in Fig. 1 (c) and Fig. 2, while $n$ even means that the unit parallelogram "overlaps" as in Fig. 3. For purposes of illustration, this very small parallelogram is magnified.

*Case n odd:*

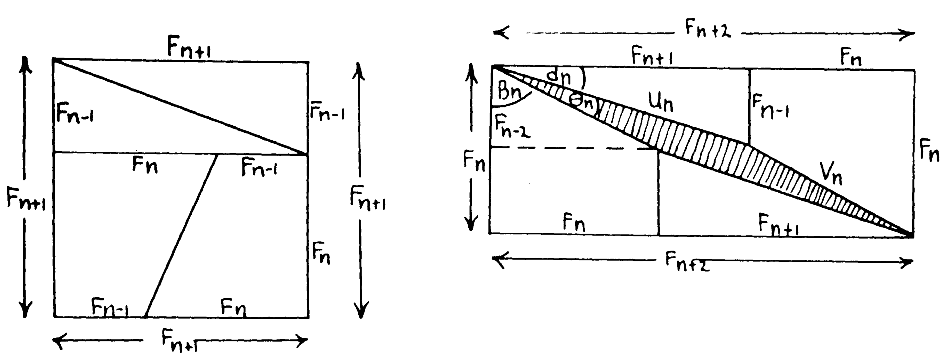



Fig. 2

Carrying out the standard construction (section 1), we obtain Fig. 2, with an obvious notation.

Then

(10) $$\theta_n = \frac{\pi}{2} - (\beta_n + \alpha_n) = \frac{\pi}{2} - \tan^{-1}\left(\frac{F_n}{F_{n-2}}\right) - \tan^{-1}\left(\frac{F_{n-1}}{F_{n+1}}\right) \quad \text{for } n \geqq 3 ,$$

i. e.,

(11) $$\theta_n = \tan^{-1}\left(\frac{F_{n-2}}{F_n}\right) - \tan^{-1}\left(\frac{F_{n-1}}{F_{n+1}}\right)$$

so that

$$\theta_n = \tan^{-1}\left(\frac{\frac{F_{n-2}}{F_n} - \frac{F_{n-1}}{F_{n+1}}}{1 + \frac{F_{n-2}}{F_n}\cdot\frac{F_{n-1}}{F_{n+1}}}\right)$$

$$= \tan^{-1}\left(\frac{F_{n-2}F_{n+1} - F_nF_{n-1}}{F_nF_{n+1} + F_{n-2}F_{n-1}}\right)$$

$$= \tan^{-1}\left(\frac{F_{n-2}(F_n + F_{n-1}) - F_{n-1}(F_{n-1} + F_{n-2})}{F_n(F_n + F_{n-1}) + F_{n-2}F_{n-1}}\right) \quad \text{using (3)}$$

$$= \tan^{-1}\left(\frac{F_{n-2}F_n - F_{n-1}^2}{F_n^2 + F_{n-1}(F_{n-1} + F_{n-2}) + F_{n-2}F_{n-1}}\right) \quad \text{using (3) again}$$

$$= \tan^{-1}\left(\frac{(-1)^{n-1}}{F_n^2 + F_{n-1}^2 + 2F_{n-2}F_{n-1}}\right) \quad \text{by (7)}$$

$$= \tan^{-1}\left(\frac{(-1)^{n-1}}{F_{2n-1} + 2F_{n-2}F_{n-1}}\right) \quad \text{by (8).}$$

Therefore

(12) $$\theta_n = \tan^{-1}\left(\frac{1}{F_{2n-1} + 2F_{n-2}F_{n-1}}\right) \quad \text{for } n \text{ odd } (\geqq 3).$$

*Case n even :*

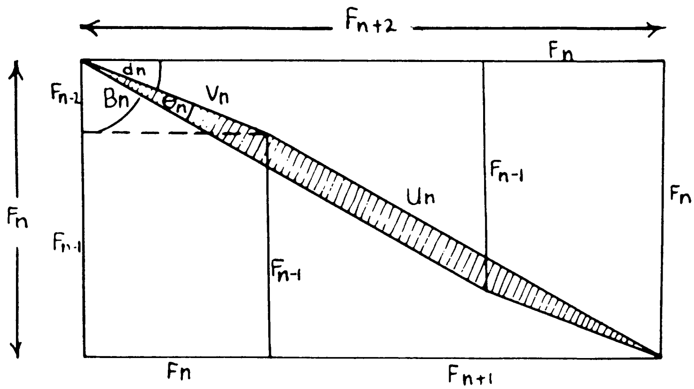



Fig. 3

In this case, the rectangle of Fig. 2 becomes that of Fig. 3. Consequently,

(13) $$\theta_n = \alpha_n + \beta_n - \frac{\pi}{2} = \tan^{-1}\left(\frac{F_{n-1}}{F_{n+1}}\right) + \tan^{-1}\left(\frac{F_n}{F_{n-2}}\right) - \frac{\pi}{2}$$

i. e.,

(14) $$\theta_n = \tan^{-1}\left(\frac{F_{n-1}}{F_{n+1}}\right) - \tan^{-1}\left(\frac{F_{n-2}}{F_n}\right)$$

i. e., the R. H. S. of (14) is minus the R. H. S. of (11). Following the calculation for $\theta_n$ through as for (12), we thus find

$$\theta_n = \tan^{-1}\left(\frac{(-1)^n}{F_{2n-1} + 2F_{n-2}F_{n-1}}\right)$$

i. e.,

(15) $$\theta_n = \tan^{-1}\left(\frac{1}{F_{2n-1} + 2F_{n-2}F_{n-1}}\right) \quad \text{when } n \text{ is even } (> 3).$$

Hence

(16) $$\theta_n = \tan^{-1}\left(\frac{1}{F_{2n-1} + 2F_{n-2}F_{n-1}}\right) \quad \text{for any } n \geqq 3,$$

$$= \tan^{-1}\left(\frac{1}{T_n}\right)$$

where

(17) $$T_n = F_{2n-1} + 2F_{n-2}F_{n-1} \, .$$

For $n < 3$, the standard construction in section 1 breaks down.

Below, in Table 1, is a list of values for $\theta_n$ for various Fibonacci triads $F_n$, $F_{n+1}$, $F_{n+2}$, beginning with $n = 3$ (it being understood that the reference is to Fig. 2 when $n$ is odd, but to Fig. 3 when $n$ is even). (Four-figure tangent tables have been used.)

Lengths of the sides of the parallelogram are

(18) $$\begin{cases} U_n = \sqrt{F_{n-1}^2 + F_{n+1}^2} \\ V_n = \sqrt{F_{n-2}^2 + F_n^2} \end{cases} \quad \text{where } U_n > V_n \, .$$

Observe that, by (16),

$$\lim_{n\to\infty}\left(\frac{T_n}{T_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{F_{2n-1} + 2F_{n-2}F_{n-1}}{F_{2n+1} + 2F_{n-1}F_n}\right)$$

$$\lim_{n\to\infty}\left(\frac{T_n}{T_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{\frac{a^{2n-1}}{\sqrt{5}} + 2\cdot\frac{a^{n-2}}{\sqrt{5}}\cdot\frac{a^{n-1}}{\sqrt{5}}}{\frac{a^{2n+1}}{\sqrt{5}} + 2\cdot\frac{a^{n-1}}{\sqrt{5}}\cdot\frac{a^{n}}{\sqrt{5}}}\right) \qquad \text{since } \lim_{n\to\infty}(b^n) = 0 \text{ in (6)}$$

$$= \lim_{n\to\infty}\left(\frac{a^{2n-3}(a^2+\frac{2}{\sqrt{5}})}{a^{2n-1}(a^2+\frac{2}{\sqrt{5}})}\right) = \frac{1}{a^2}$$

$$= 1 - \frac{1}{a} \qquad \text{since } a^2 - a - 1 = 0 \text{ from (6)}$$

(19) $= 1 - .618\ldots = .382\ldots$ .

Table 1

| $n$ | Fibonacci triad | | | $\theta_n$ | $V_n, U_n$ |
|---|---|---|---|---|---|
| 3 | 2 | 3 | 5 | $\tan^{-1}(\frac{1}{7}) = \tan^{-1}(.1429) = 8°8'$ | $\sqrt{5}, \sqrt{10}$ |
| 4 | 3 | 5 | 8 | $\tan^{-1}(\frac{1}{17}) = \tan^{-1}(.0588) = 3°22'$ | $\sqrt{10}, \sqrt{29}$ |
| 5 | 5 | 8 | 13 | $\tan^{-1}(\frac{1}{46}) = \tan^{-1}(.0217) = 1°15'$ | $\sqrt{29}, \sqrt{73}$ |
| 6 | 8 | 13 | 21 | $\tan^{-1}(\frac{1}{119}) = \tan^{-1}(.0084) = 29'$ | $\sqrt{73}, \sqrt{194}$ |
| 7 | 13 | 21 | 34 | $\tan^{-1}(\frac{1}{313}) = \tan^{-1}(.0032) = 11'$ | $\sqrt{194}, \sqrt{505}$ |
| 8 | 21 | 34 | 55 | $\tan^{-1}(\frac{1}{818}) = \tan^{-1}(.0012) = 4'$ | $\sqrt{505}, \sqrt{1325}$ |
| 9 | 34 | 55 | 89 | $\tan^{-1}(\frac{1}{2143}) = \tan^{-1}(.0005) = 2'$ | $\sqrt{1325}, \sqrt{3506}$ |
| 10 | 55 | 89 | 144 | $\tan^{-1}(\frac{1}{5609}) = \tan^{-1}(.0002) = 1'$ | $\sqrt{3506}, \sqrt{9077}$ |

The sequence $(T_n/T_{n+1})$ converges fairly rapidly to its limit; e.g., for $n = 9$ the limit is correct to three decimal places.

Likewise, the limiting ratio of the lengths of the sides of the parallelogram (shorter to longer) is found to be

$$\lim_{n\to\infty}\left(\frac{V_n}{U_n}\right) = \lim_{n\to\infty}\sqrt{\frac{F_{n-2}^2 + F_n^2}{F_{n-1}^2+F_{n+1}^2}}$$

(20) $$= \frac{1}{a} \text{ on calculation}$$

$$= \lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}}\right).$$

This is not completely unexpected for, as $n \to \infty$, $\theta_n \to 0$, and then

$$V_n \to -b\sqrt{3}\,F_n\,, \quad U_n \to -b\sqrt{3}\,F_{n+1}\,,$$

as may be verified using (4).

**4. Generalization of the paradox.** Instead of (1), suppose we consider the *generalized Fibonacci sequence*

(21) $H_{pq}$:

| $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ | $H_9$ | ... |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p+q$ | $2p+q$ | $3p+2q$ | $5p+3q$ | $8p+5q$ | $13p+8q$ | $21p+13q$ | $34p+21q$ | ... |

defined by

(22) $$H_1 = p\,, \quad H_2 = p+q$$

(23) $$H_{n+2} = H_{n+1} + H_n \quad \text{(recurrence relation)}$$

where $p$, $q$ are arbitrary integers.

Assuming a solution of the form $H_n = \alpha A^n + \beta B^n$ and employing the usual method for difference equations and using (22), we obtain as the *n th generalized Fibonacci number*

(24) $$H_n = \frac{1}{2\sqrt{5}}(la^n - mb^n) = pF_n + qF_{n-1} \quad \text{also}$$

where

(25) $$\begin{cases} l = 2(p-qb) \\ m = 2(p-qa) \end{cases}$$

so that

(26) $$\frac{lm}{4} = p^2 - pq - q^2 = e\ .$$

Note that (after simplification)

(27) $$\lim_{n\to\infty}\left(\frac{H_n}{H_{n+1}}\right) = \frac{1}{a}\,,$$

corresponding to (9).

The symbol $e$ is the same as that used by Tagiuri (quoted in Dickson [2]), who did somewhat similar work, though in a slightly different notation. Shortly, we shall obtain a geometrical meaning for $e$. Before this, however, we need to derive a few results for $H_{pq}$ corresponding to (7) and (8) for $f$. Of course, the classical Fibonacci sequence is a special case of our generalized sequence when $p = 1$, $q = 0$.

Elsewhere [3], I have given several formulae for the generalized Fibonacci sequence, but here we are concerned only with those relating to the paradox.

Corresponding to (7) is

*Theorem 1:*

$$H_n H_{n+2} - H_{n+1}^2 = (-1)^{n+1} e \; . \tag{28}$$

*Proof:* By (24), we have

$$\begin{aligned} H_n H_{n+2} - H_{n+1}^2 &= \frac{1}{2\sqrt{5}}(la^n - mb^n) \cdot \frac{1}{2\sqrt{5}}(a^{n+2} - mb^{n+2}) - [\frac{1}{2\sqrt{5}}(la^{n+1} - mb^{n+1})]^2 \\ &= \frac{1}{20}\{l^2(a^{2n+2} - a^{2n+2}) + m^2(b^{2n+2} - b^{2n+2}) \\ &\qquad - lm(a^n b^{n+2} + a^{n+2} b^n - 2a^{n+1} b^{n+1})\} \\ &= -\frac{lm}{20}(ab)^n (a^2 + b^2 - 2ab) \\ &= -(-1)^n \frac{lm}{20}(a-b)^2 \qquad \text{by (6)} \\ &= (-1)^{n+1}\frac{lm}{4} \qquad \text{by (6)} \\ &= (-1)^{n+1} e \qquad \text{by (26)} \; . \end{aligned}$$

Corresponding to (8) is

*Theorem 2:*

$$H_{n-1}^2 + H_n^2 = (2p - q)H_{2n-1} - eF_{2n-1} \qquad (n \geqq 2) \; . \tag{29}$$

*Proof:* Again by (24), we have

$$\begin{aligned} &H_{n-1}^2 + H_n^2 - (2p-q)H_{2n-1} \\ &= \frac{1}{20}\{(la^{n-1} - mb^{n-1})^2 + (la^n - mb^n)^2\} - \frac{l+m}{2} \cdot \frac{1}{2\sqrt{5}}(la^{2n-1} - mb^{2n-1}) \quad \text{by (25)} \\ &= \frac{1}{4\sqrt{5}}\left\{\frac{l^2(a^{2n-2} + a^{2n} - 5a^{2n-1})}{\sqrt{5}} + \frac{m^2(b^{2n-2} + b^{2n} + \sqrt{5}b^{2n-1})}{\sqrt{5}}\right. \\ &\qquad \left. - \frac{lm(2(a^{n-1}b^{n-1} + a^n b^n) + \sqrt{5}(a^{2n-1} - b^{2n-1}))}{\sqrt{5}}\right\} \end{aligned}$$

$$H_{n-1}^2 + H_n^2 - (2p-q)H_{2n-1}$$

$$= \frac{1}{4\sqrt{5}}\left\{\frac{l^2a^{2n-2}(1-\sqrt{5}\,a+a^2)}{\sqrt{5}} + \frac{m^2b^{2n-2}(1+\sqrt{5}\,b+b^2)}{\sqrt{5}} - \frac{lm(2a^{n-1}b^{n-1}(1+ab)+\sqrt{5}\,(a^{2n-1}-b^{2n-1}))}{\sqrt{5}}\right\}$$

$$= -\frac{lm}{4\sqrt{5}}(a^{2n-1}-b^{2n-1}) \qquad \text{since} \begin{cases} 1-\sqrt{5}\,a+a^2 = 0 \\ 1+\sqrt{5}\,b+b^2 = 0 \text{ by (5)} \\ 1+ab = 0 \end{cases}$$

$$= -\frac{lm}{4}\cdot\frac{a^{2n-1}-b^{2n-1}}{\sqrt{5}}$$

$$= -eF_{2n-1} \qquad \text{by (4) and (26)}.$$

Imagine now that Figs. 2 and 3 have been generalized so that each Fibonacci number $F_i$ is replaced by the corresponding generalized Fibonacci number $H_i$ in the generalized sequence $H_{pq}$. Following through the calculations leading to (12), (15) and (16), we obtain, by virtue of Theorems 1 and 2, that the angle of the generalized parallelogram is given by

$$(30) \qquad \theta_n = \tan^{-1}\left(\frac{e}{(2p-q)H_{2n-1} - eF_{2n-1} + 2H_{n-2}H_{n-1}}\right) \quad \text{for any } n \ (\geq 3)$$

$$= \tan^{-1}\left(\frac{1}{S_n}\right)$$

where

$$(31) \qquad S_n = \frac{(2p-q)H_{2n-1} - eF_{2n-1} + 2H_{n-2}H_{n-1}}{e} \quad \text{for } n \geq 3.$$

Lengths of sides of the generalized parallelogram are

$$(32) \qquad \begin{cases} W_n = \sqrt{H_{n-1}^2 + H_{n+1}^2} \\ X_n = \sqrt{H_{n-2}^2 + H_n^2} \end{cases} \quad \text{where } W_n > X_n.$$

Theorem 1, by analogy with the geometric meaning of (7), shows that $e$ is the area of the generalized parallelogram. Thus, we have found a geometrical interpretation of $e$ (a number which plays a dominant role in formulae in the generalized theory, as, e. g., in Theorems 1 and 2).

Paralleling results (19) and (20) in section 3, we discover that

$$(33) \qquad \lim_{n\to\infty}\left(\frac{S_n}{S_{n+1}}\right) = \frac{1}{a^2};$$

$$(34) \qquad \lim_{n\to\infty}\left(\frac{X_n}{W_n}\right) = \frac{1}{a} = \lim_{n\to\infty}\left(\frac{H_n}{H_{n+1}}\right).$$

Golden section is therefore, as expected, involved in the generalized theory. [In (33), it may be noted that the cancelling factor, corresponding to $a^2+(2/\sqrt{5})$ in the work leading to (19), is $l^2(a^2+(2/\sqrt{5}))$.]

It now remains for this general theory to be made explicit by a few numerical examples. Table 2 enumerates values of $e$ (i. e., areas of parallelograms) in some particular Fibonacci sequences, $e$ being constant for all values of $n$ in any particular sequence. Several different sequences may possess the same value for $e$.

Table 2

| $p$ | $q$ | $e$ | Sequence | Elements of Sequence | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ |
| 1 | 0 | 1 | $H_{10}$ (= $f$) | 1 | 1 | 2 | 3 | 5 | 8 | 13 |
| 2 | 1 | 1 | $H_{21}$ | 2 | 3 | 5 | 8 | 13 | 21 | 34 |
| 3 | 1 | 5 | $H_{31}$ | 3 | 4 | 7 | 11 | 18 | 29 | 47 |
| 4 | 1 | 11 | $H_{41}$ | 4 | 5 | 9 | 14 | 23 | 37 | 60 |
| 4 | 2 | 4 | $H_{42}$ (= $2H_{21}$) | 4 | 6 | 10 | 16 | 26 | 42 | 68 |
| 5 | 1 | 19 | $H_{51}$ | 5 | 6 | 11 | 17 | 28 | 45 | 73 |
| 5 | 2 | 11 | $H_{52}$ | 5 | 7 | 12 | 19 | 31 | 50 | 81 |
| 5 | 3 | 1 | $H_{53}$ | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

A simple computation shows that the total number of sequences $H_{pq}$ for $0 \leqq p \leqq n$, $0 \leqq q \leqq n$ is $n+2(1+2+\cdots+n) = n(n+2)$. ($p=q=0$ excluded.)

To illustrate the mechanism of a particular sequence, we choose $H_{52}$ for which $2p-q = 8$, $e = 11$, and parallel the results of Table 1. From (21), the sequence $H_{52}$ (partially set out in Table 2) is:

$$(35)\quad H_{52}: \begin{array}{cccccccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} & \cdots \\ 5 & 7 & 12 & 19 & 31 & 50 & 81 & 131 & 212 & 343 & 555 & \cdots \end{array}$$

and the area of the corresponding parallelogram is 11 sq. units.

Clearly, $\theta_n \to 0$ more rapidly in this sequence than in the classical sequence $f$. This is because elements of $H_{52}$ are increasingly greater than corresponding elements of $f$, for, as we may readily calculate,

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{H_n}{F_n}\right) &= \frac{l}{2} \\ &= p-qb \qquad \text{for } H_{pq} \\ &= 4+\sqrt{5} \qquad \text{for } H_{52} \text{ from (5)} . \end{aligned}$$

Table 3

| $n$ | Fibonacci triad | $\theta_n$ | $X_n, W_n$ |
|---|---|---|---|
| 3 | 12 19 31 | $\tan^{-1}(\frac{11}{263}) = \tan^{-1}(.0418) = 2°24'$ | $\sqrt{169}, \sqrt{410}$ |
| 4 | 19 31 50 | $\tan^{-1}(\frac{11}{673}) = \tan^{-1}(.0163) = 56'$ | $\sqrt{410}, \sqrt{1105}$ |
| 5 | 31 50 81 | $\tan^{-1}(\frac{11}{1778}) = \tan^{-1}(.0062) = 21'$ | $\sqrt{1105}, \sqrt{2861}$ |
| 6 | 50 81 131 | $\tan^{-1}(\frac{11}{4639}) = \tan^{-1}(.0024) = 8'$ | $\sqrt{2861}, \sqrt{7522}$ |
| 7 | 81 131 212 | $\tan^{-1}(\frac{11}{12161}) = \tan^{-1}(.0009) = 3'$ | $\sqrt{7522}, \sqrt{19661}$ |

One interesting question is this: Are there any sequences for which $\lim_{n\to\infty}(T_{n+r}/S_n) = 1$? Using the formulae (17) and (31), we find, after calculation, that this is true if

$$ea^{2r}(a^2+\frac{2}{\sqrt{5}}) = a^2[(2p-q)\frac{l}{2}-e]+\frac{l^2}{2\sqrt{5}}$$

whence it follows that if $r = 1$. $2p-q = e$, $l = 2a\sqrt{e}$, then $e = 5$ and $p = 3$, $q = 1$. That is, a solution occurs in the case of the sequence $H_{31}$ in which

$$\lim_{n\to\infty}\left(\frac{T_{n+1}}{S_n}\right) = 1, \quad \text{i.e.,} \quad \theta_{n+1} \text{ (for } f) \to \theta_n \text{ (for } H_{31}).$$

Originally, I chose $H_{31}$ as my illustration (Table 3) but discarded it because of this peculiarity. The crux of the matter is that $2p-q = e$ which allows simplification in (31). [Another sequence for which $2p-q = e$ $(= 4)$ is $H_{20}$ but this is merely $2f$, and in this case $\lim_{n\to\infty}(T_n/S_n) = 1$.]

A sequence related to $H_{31}$ is $H_{74}$ which is the same as $H_{31}$ except that the first two elements are missing. For $H_{74}$ we have $2p-q = 10 = 2e$ with $l = 2(1+2a)\sqrt{e}$ and, of course, $\lim_{n\to\infty}(T_{n+3}/S_n) = 1$. Infinitely many similarly related sequences exist.

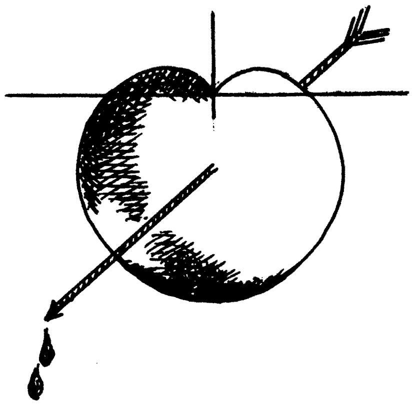
Finally, we comment on the fact that $e$ may have negative values, a result which does not affect our work on the parallelograms. [Obviously, $e$ can never be zero for this would imply, by (26), that $l = 0$ or $m = 0$, i.e. that $p = qb$ or $qa$. But $b$ and $a$ are irrational, while $p$, $q$ are integers.] If $e > 0$, then assuming $p > 0$, $q > 0$, we must have $(p/q) > a$ whereas for $e < 0$ we have $(p/q) < a$, keeping in mind that $a = 1.61\cdots$.

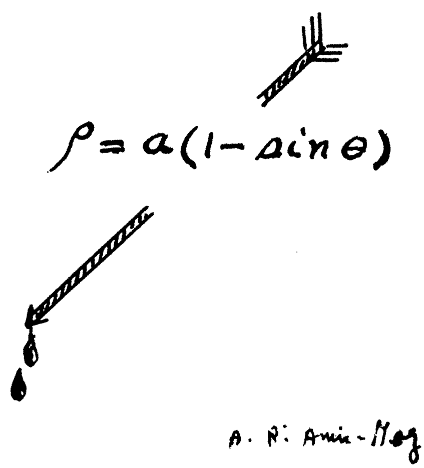
Many new problems have been raised in the last few paragraphs, but these are beyond the scope of the problems we proposed to solve.

### *REFERENCES*


1. Ball, W. W. R., *Mathematical Recreations and Essays*, 1931, pp. 52-54.
2. Dickson, L. E., *History of the Theory of Numbers*, Vol. 1, 1952, pp. 393-407. (For references to Tagiuri, consult p. 404.)
3. Horadam, A. F., *Generalised Fibonacci Sequence*, (to be published in Amer. Math. Monthly).


---






---

# AN APPLICATION OF SCHWARZ'S INEQUALITY TO CURVE FITTING

DMITRI THORO, San Jose State College

Consider

(1) $$S = \sum (y_i - a_0 - a_1 x_i)^2 \,, \quad \text{where } \sum = \sum_{i=1}^{n} \,.$$

The straight line $y = a_0 + a_1 x$ best fitting a set of $n$ points $(x_i, y_i)$ in the least squares sense is obtained by finding $a_k$ $(k = 0, 1)$ such that $S$ is a minimum.

From the *necessary* condition $\frac{\partial S}{\partial a_0} = \frac{\partial S}{\partial a_1} = 0$ one obtains the *normal equations*:

(2) $$\sum y_i = n a_0 + a_1 \sum x_i$$

(3) $$\sum x_i y_i = a_0 \sum x_i + a_1 \sum x_i^2 \,.$$

The purpose of this note is to show that the above procedure actually yields the minimum value of $S$.

By the usual test for extrema, a sufficient condition is that $\frac{\partial S}{\partial a_0} = \frac{\partial S}{\partial a_1} = 0$ and, for the values of $a_k$ which make these partial derivatives vanish,

$$\frac{\partial^2 S}{\partial a_0^2} > 0 \quad \text{and} \quad D \equiv \begin{vmatrix} \dfrac{\partial^2 S}{\partial a_0^2} & \dfrac{\partial^2 S}{\partial a_0 \partial a_1} \\ \dfrac{\partial^2 S}{\partial a_0 \partial a_1} & \dfrac{\partial^2 S}{\partial a_1^2} \end{vmatrix} > 0 \,.$$

Thus

$$D = \begin{vmatrix} 2n & 2\Sigma x_i \\ 2\Sigma x_i & 2\Sigma x_i^2 \end{vmatrix} = 4\left(n \sum x_i^2 - \left(\sum x_i\right)^2\right) .$$

But from Schwarz's inequality $(\Sigma u_i^2)(\Sigma v_i^2) \geq (\Sigma u_i v_i)^2$ (setting $u_i = x_i$, $v_i = 1$ and assuming that all the $x_i$ are not equal) we have

(4) $$n \sum x_i^2 - \left(\sum x_i\right)^2 > 0 \,.$$

Moreover, (4) guarantees that the normal equations will have a unique solution since $D/4$ is the determinant of the coefficients.

# THE TWO MOST ORIGINAL CREATIONS OF THE HUMAN SPIRIT

ELMER B. MODE, Emeritus, Boston University

"The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music."

A. N. Whitehead, *Science and the Modern World.*

**1. Introduction.** In the quotation given above a great Anglo-American philosopher [1] characterized two distinct fields of human interest, one a science, the other an art. The arts and the sciences, however, are not mutually exclusive. Art has often borrowed from science in its attempts to solve its problems and to perfect its achievements. Science in its higher forms has many of the attributes of an art. Vivid aesthetic feelings are not at all foreign in the work of the scientist. The late Professor George Birkhoff, in fact, wrote as follows:

"A system of laws may be beautiful, or a mathematical proof may be elegant, although no auditory or visual experience is directly involved in either case. It would seem indeed that all feeling of desirability which is more than mere appetite has some claim to be regarded as aesthetic feeling." [2]

Serge Koussevitzky, noted conductor, has stated also that "there exists a profound unity between science and art." [3]

It is not, however, the purpose of this paper, to discuss the relationships between the sciences and the arts, but rather to enumerate some of the lesser known attributes which music and mathematics have in common. There is no attempt to establish a thesis.

**2. Number and Pitch.** The study of mathematics usually begins with the natural numbers or positive integers. Their symbolic representation has been effectively accomplished by means of a radix or scale of ten, the principle of place-value where the position of a digit indicates the power of ten to be multiplied by it, and a zero. The concept of number is most basic in mathematics. We cannot directly sense number. A cardinal number, such as five, is an abstraction which comes to us from many concrete instances each of which possesses other attributes not even remotely connected with the one upon which our interest is fixed. Such widely differing groups as the fingers of the hand, the sides of the pentagon, the arms of a starfish, and the Dionne quintuplets, are all instances of "fiveness," the property which enables each group to be matched or placed into one-to-one correspondence with the other. The establishment of such equivalence requires no knowledge of mathematics, only good eyesight. With these facts in mind we may state a definition familiar to mathematicians. *The (cardinal) number of a group of objects is the invariant property of the group and all other groups which can be matched with it.*

The positive integers constitute, however, but a small portion of the numbers of mathematics. The former mark off natural intervals in the continuum of real numbers. The difference between two small groups of objects is readily sensed; man finds no difficulty in distinguishing *visually,* at once, between *three* and *four* objects, but the distinction between, say, thirty-two and thirty-three objects calls for something more than good vision.

In music, study begins with notes or tones. In western music their symbolic representation is accomplished by means of a scale of seven, a principle of position, and the rest, which denotes cessation of tone. There is something permanent and unchangeable about a given note. You may sing it, the violin string may emit it, the clarinet may sound it, and the trumpet may fill the room with it. The quality or timbre, the loudness or intensity, and the duration of one sound may be markedly different from another; yet among these differences of sound there remains one unchanging attribute, its pitch. This is the same for a single such note or any combination of them. The pitch of a note may then be defined as *the invariant property of the note and all other notes which may be matched with it.* Notes which can be matched are said to be in unison. Pitch, also, is an abstraction, derived from many auditory experiences. The establishment of pitch equivalence does not require a knowledge of music, only a keen ear.

The notes of the diatonic scale mark off convenient intervals in a continuum of pitches. Within a given range, the interval between two tones of the scale is, in general, readily sensed, but outside of such a range the human ear may fail to distinguish between or even to hear two differing tones. As a matter of fact, "tones" removed from the range of audibility cease to be such. As psychological entities they disappear and may be identified only as vibrations in a physical medium.

Invariance of pitch is an important musical property and the ability of a musician not playing a keyed instrument to maintain this property for a given note is a necessary, but not a sufficient condition for his artistry. This recalls the story of the distracted singing teacher who, after accompanying his none-too-apt pupil, sprang suddenly from the piano, thrust his fingers wildly through his hair, and shouted, "I play the white notes, and I play the black notes, but you sing in the cracks."

**3. Symbols.** Mathematics is characterized by an extensive use of symbols. They are indispensable tools in the work; they constitute the principal vehicle for the precise expression of ideas; without them modern mathematics would be non-existent. The most important mathematical symbols are, with few exceptions, in universal use among the civilized countries of the world.

Music also is distinguished by a universal symbolism. The creation of anything but the simplest musical composition or the transmission of significant musical ideas is difficult if not impossible without the symbols of music.

Incidentally it may be remarked that the page of a musical score and the page of a book in calculus are equally unintelligible to the uninitiated.

There are few fields of activity outside of mathematics (including logic) and music which have developed so extensively their own symbolic language. Chemistry and phonetics are nearest in this respect.

In both music and mathematics preliminary training involves the acquiring of technique. Mathematics demands such facile manipulation of symbols that the detailed operations become mechanical. We are encouraged to eliminate the necessity for elementary thinking as much as possible, once the fundamental logic is made plain. This clears the way for more complicated processes of reasoning.

"It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them." [4]

In music also, the preliminary training involves a learning of technique. The aim here is to be able to read, or to write, or to translate into the appropriate physical actions, notes and combinations of them with such mechanical perfection that the mind is free for the creation and the interpretation of more profound musical ideas.

**4. Logical Structure.** The framework of a mathematical science is well known. We select a class of objects and a set of relations concerning them. Some of these relations are assumed and others are deduced. In other words, from our axioms and postulates we deduce theorems embracing important properties of the objects involved.

Music likewise has its logical structure. The class of objects consists of such musical elements as tones, intervals, progressions, and rests, and various relations among these elements. In fact, the structure of music has been formally described as a set of postulates according to the customary procedure of mathematical logic. [5]

In mathematics a development is carried forward according to the axioms or postulates. If these are obeyed the results are correct, in the mathematical sense, although they may not be interesting or useful. Mere obedience to law does not create an original piece of mathematical work. This requires technical skill, imagination, and usually a definite objective.

Music also has its axioms or laws. These may be as simple as the most obvious things in elementary mathematics – the whole equals the sum of all its parts – if we are counting beats in a measure; they may be less obvious to the layman, such as the canons of harmony or the structural laws of a classical symphony. Here again we may follow the laws of music scrupulously without ever creating a worth-while bit of original music. Technical skill, imagination, the fortunate mood, and usually a definite objective are requisites for the creation of a composition which not only exhibits obedience to musical laws but expresses significant ideas also. Occasionally the musician becomes bold and violates the traditional musical axioms so that the resulting effects may at first sound strange or unpleasant. These may become as useful, provoking, and enjoyable, as a non-Euclidean geometry or a non-Aristotelian logic. In such manner did Wagner, Debussy, Stravinsky, and others extend the bounds of musical thought. In mathematics as well as in music one may have to become accustomed to novel developments before one learns to like them.

Benjamin Peirce defined mathematics as "the science which draws necessary conclusions." The operations from hypothesis to theorem proceed in logical order without logical hesitation or error. When the series of deductive operations flows swiftly and naturally to its inevitable conclusion, the mathematical structure gives a sense of satisfaction, beauty, and completeness. Sullivan characterizes the opening theme of Beethoven's Fifth Symphony as one which "immediately, in its ominous and arresting quality, throws the mind into a certain state of expectance, a state where a large number of happenings belonging to a certain class, can logically follow." [6] The same is true of the opening phrase of the prelude to *Tristan and Isolde*, or of any really great enduring masterpiece.

An interesting departure from the usual logical structure of a musical composition occurs in the Symphonic Variations, "Istar" by Vincent d'Indy. Instead of the initial announcement of the musical theme with its subsequent variations, "the seven variations proceed from the point of complex ornamentation to the final stage of bare thematic simplicity." Philip Hale, the eminent Boston musical critic related the following anecdote in the Boston Symphony Bulletin of April 23, 1937.

"M. Lambinet, a professor at a Bordeaux public school, chose in 1905 the text 'Pro Musica' for his prize-day speech. He told the boys that the first thing the study of music would teach them would be logic. In symphonic development logic plays as great a part as sentiment. The theme is a species of axiom, full of musical truth, whence proceed deductions. The musician deals with sounds as the geometrician with lines and the dialectitian with arguments. The master went on to remark: 'A great modern composer, M. Vincent d'Indy, has reversed the customary process in his symphonic poem "Istar." He by degrees unfolds from initial complexity the simple idea which was wrapped up therein and appears only at the close, like Isis unveiled, like a scientific law discovered and formulated.' The speaker found this happy definition for such a musical work – 'an inductive symphony.' "

**5. Meaning.** A mathematical formula represents a peculiarly succinct and accurate representation of meaning which cannot be duplicated by any other means. It is concerned with the phenomenon of variability; it involves the function concept. "A mathematical formula can never tell us what a thing is, but only how it behaves." [7]

How true this is of music! A theme of great music compresses into a small interval of space or time, inimitably and accurately, a remarkable wealth of meaning. Music is not fundamentally concerned with the description of static physical objects, but with the impressions they leave under varying aspects. Debussy's "La Mer" is a fine example of this type of description. Music's interest is often not in the physical man but in his changing moods, in his emotions. One of the sources of the greatness of "Die Walküre" is Wagner's genius for portraying vividly the conflicting aspects of Wotan's nature—as god and as man.

The meaning of musical motive grows with study. It is usually exploited or developed and from it are derived new figures of musical expression. A good theme demands more than the casual hearing before its deep significance is completely appreciated. It is often worked up from an entirely insignificant motive as in Beethoven's Fifth Symphony or in Mozart's G minor symphony. In mathematics a basic formula or equation may have implications which can be understood only after much study. It may appear to be almost trivial as in the case of $a+b=b+a$ or it may be less obvious and more elegant as in the case of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 .$$

Music consists of abstractions, and at its best gives expression to concepts which represent the most universal features of life. Beethoven's music expresses powerfully the great aspirations, struggles, joys, and tragedies of human existence. The Eroica symphony may have been composed with Napoleon in mind but it portrays far more than the career of a single man. It is a portrayal of the heroic in man and as such is universal in its application. It is well known that a musical passage or composition may produce different responses among people. The possibility of varying interpretation constitutes one of the sources of music's uniqueness and a reason for its power. It is an evidence of its universality. Herein lies a fundamental difference between music and painting or sculpture. The effect of a musical episode is due to its wide potential emotional applicability; the effect of a painting or piece of sculpture is due to its concreteness. Attempts at abstract representations by painters have not been generally successful; attempts at stark realism in music have likewise failed. Music in its most abstract form, as for example, Bach's or Mozart's, often defies application to the concrete. It seems to be above mundane things, in the realm of pure spirit.

So it is with mathematics. Our conclusions are always abstract, and universal in their application, although they may have originated from a special problem. The possibilities of interpretation and application of a given theorem or formula are unlimited. Poincaré is reported to have said that even the same mathematical theorem has not the same meaning for two different mathematicians. What differing reactions may ensue when Laplace's equation is set up before an audience of mathematicians! What differing degrees of abstractness are suggested by the two equations previously written!

**6. The Creative Process.** "It is worth noting ... that it is only in mathematics and music that we have the creative infant prodigy; ... the boy mathematician or musician, unlike other artists, is not utilizing a store of impressions, emotional or other, drawn from experience or learning; he is utilizing inner resources. ..." [8]

Statements of this type have led many to believe that mathematical talent and musical talent have more than an accidental relation. Some feel that mathematicians are more naturally drawn to music than musicians are to mathematics. As far as the writer has been able to ascertain, no serious investigations on the relation between the two talents have been published. A brief study of exceptionally gifted children yields no testimony that the child prodigy in music has more than the average mathematical sense, or that the child prodigy in mathematics has exceptional ability in music.

In a recent article, *Mind and Music,* [9] the inimitable English music critic, Ernest Newman, discusses the role that the subconscious mind might play in the creative processes of music. Hampered by a dearth of reliable testimony on this subject, he attempts, nevertheless, to estimate this role. Berlioz and Wagner had written of their creative experiences without attempting any self-analysis. So also had Mozart although Newman does not refer to him. Newman feels that the Memoirs of Hector Berlioz are not too reliable in this respect. Wagner's letters, however, seem to indicate that many of his musical ideas were the result of an upsurge from the unconscious depths of his mind of ideas long hidden but suddenly crystallizing. The activity of his conscious mind was often displaced by the upward thrust of these latent creative forces.

The interpretation thus suggested is strikingly similar in many respects to that described by Jacques Hadamard in his *Essay on the Psychology of Invention in the Mathematical Field.* [10] This noted mathematician draws on the related experiences of Poincaré, Helmholtz, Gauss, and others to discuss the origin of the inspiration or sudden insight that contributes to, completes, or initiates an original work. The role of thoughts that lie vague and undiscernible in the subconscious, only to become, of a sudden, clear and discernible after a period of unsuspected incubation, is described in undogmatic terms. One cannot affirm, of course, that these opinions concerning the creative process are confined solely to music and mathematics, but it is interesting that they are voiced by two eminent scholars, one from each field.

The greatest works of music are distinguished by their intellectual content as well as by their emotional appeal. The sacred music of Bach, the symphonies of Beethoven, or the operas of Wagner, offer subjects for analysis and discussion, as well as opportunities for emotional experience. Each composer had ideas to "work out," ideas to be developed and clarified by the forms and artifices of music, the object being to make their full significance felt by the appreciative listener.

Mathematical creativity involves very much the same general development. Concepts must be clarified, operations carried out, latent meanings revealed. If these are significant and logically developed the result has a unity and a sense of completeness which brings intellectual and aesthetic satisfaction to both author and reader.

**7. Aesthetic Considerations.** To many, mathematics seems to be a forbidding subject. Its form seems to be more like that of a skeleton than that of a living, breathing, human body. This idea, is, of course, derived from its abstract character and from the demands which it makes for sharply defined concepts, terse methods of expression, and precise rules of operation. In a sense, mathematics lacks richness, if by richness we mean the presence of those impurities which impart savor and color. These impurities may be in the nature of concrete examples, illustrations from, or applications to fields other than mathematics. They may represent departures from the normal abstract logical development, and may make no contributions whatsoever to the formal structure which constitutes mathematics. But if we subtract from the richness of mathematics, we add also to its purity, for in mathematics the structure or form is more important than its applications. We may apply the mathematics to many problems associated with human existence, but these applications are not essential parts of pure mathematics, they lie apart from it.

"In music the flavor of beauty is purest, but because it is purest it is also least rich. ... A melody is a pure form. Its content is its form and its form is its content. A change in one means a change in other. We can, of course, force an external content upon it, read into it stories or pictures. But when we do so we know that they are extraneous and not inherent in the music." [11]

In a different sense mathematics is over-rich for its fields are unlimited in extent and fertility.

"But no one can traverse the realm of the multiple fields of modern Mathematics and not realize that it deals with a world of its own creation, in which there are strangely beautiful flowers, unlike anything to be found in the world of external entities, intricate structures with a life of their own, different from anything in the realm of natural science, even new and fascinating laws of logic, methods of drawing conclusions more powerful than those we depend upon, and ideal categories very widely different from those we cherish most." [12]

One needs here but to change a few words in order to describe the unique and lovely creations of music. The melodies and harmonies of music are its own inventions. They are often mysteriously beautiful, incapable of description by other means and without counterpart elsewhere in the world about us. A musical composition may be of the utmost simplicity or of the most intricate character, yet it may "well-nigh express the inexpressible." It is exactly this ability to convey the "inexpressible" ideas that give mathematics and music much in common. The mathematics student who seeks always a meaning or picture of each new proposition often fails to appreciate the power of that which defies representation.

**8. Conclusion.** There is much of interest to those who love both music and mathematics, and much has been written by mathematicians on the bearings of one field on the other. Archibald has written delightfully of some of their human aspects as well as the scientific. Birkhoff has attempted the evaluation of musical aesthetics by quantitative methods. Miller and others have brought the instruments of physics to bear upon the problems of musical tone and acoustics.

Success in music and in mathematics also depend upon very much the same things — fine technical equipment, unerring precision, and abundant imagination, a keen sense of values, and, above all, a love for truth and beauty.

## *REFERENCES*

1. Whitehead, A. N., *Science and the Modern World*, New York, 1925, p. 29.
2. Birkhoff, G. D., *Aesthetic Measure*, Cambridge, 1933, p. 209.
3. At a symphony concert given in honor of the American Association for the Advancement of Science, Boston, Mass., Dec. 27, 1933.
4. Whitehead, A. N., *An Introduction to Mathematics*, New York, 1911, p. 61.
5. Langer, S. K., *A Set of Postulates for the Logical Structure of Music*, Monist, Vol. 39, 1929, pp. 561-570.
6. Sullivan, J. W. N., *Aspects of Science*, London, 1923, p. 167.
7. Jeans, J. H., *The Mysterious Universe*, Cambridge, 1930, pp. 151, 152.
8. See 6, p. 168.
9. Newman, E., *Mind and Music*, Sunday Times (London), July 8, 1956.
10. Hadamard, J., *An Essay on the Psychology of Invention in the Mathematical Field,* Princeton, 1945.
11. Schoen, M., *Art and Beauty*, New York, 1932, pp. 190-192.
12. Shaw, J. B., *Cause, Purpose, Creativity*, Monist, Vol. 33, 1923, p. 355.

---

## Maybe

*Within your lifetime will, perhaps,*
*As souvenirs from distant suns*
*Be carried back to earth some maps*
*Of planets and you'll find that one's*
*So hard to color that you've got*
*To use five crayons. Maybe, not.*

Marlow Sholander

---

# POLYNOMIAL SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS BY DIFFERENTIATION

CLEMENT EARL FALBO, University of Texas

**Foreword.** This paper introduces differentiation as a method for finding particular solutions to some non-homogeneous linear differential equations. The types of differential equations solved by this method have polynomial solutions. By excluding zero as a polynomial, the uniqueness of the polynomial solution to each of these equations can be proved. A theorem which applies differentiation as a method of solution is given.

**1. Introduction.** *1.1 Notation.* Wherever possible, the symbolism in this paper follows the usual convention, e. g. $x$ denotes a real number, $y(x)$ denotes the second number of the ordered pair $(x, y)$, and $y$ is used to denote a single-valued function of one variable.

The notation for derivatives will be as follows:

$$D_x y = y^{[1]}, \quad \text{and} \quad D_x y^{[n-1]} = y^{[n]}, \quad \text{where } n = 2, 3, \cdots .$$

A polynomial of degree $n$ will be denoted by $P_n(x)$, i. e.,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,$$

where $a_i$ is a real number for $i = 0, 1, \cdots, n$. Note that zero is excluded as a polynomial.

To facilitate the definition of the method of solution to be used, the symbol, $F(y, y^{[i]})$, will denote a linear function of $y, y^{[1]}, y^{[2]}, \cdots, y^{[n]}$. The coefficients of $y^{[i]}$ in $F$ will be either constants, or polynomials in $x$, and the coefficient of $y$ is one.

*1.2 Type of equation.* The theorems in this paper concern finding particular solutions, by the use of differentiation, for certain cases of the ordinary differential equation,

$$F(y, y^{[i]}) = P_n(x) .$$

*1.3 Method of solution. Definition:* The statement that, "a polynomial solution, $R_n(x)$ of the differential equation

$$F(y, y^{[i]}) = P_n(x) \tag{1}$$

can be obtained by the method of differentiation," means:

1. There exists a polynomial $R_n(x)$ of degree $n$ such that $y = R_n(x)$ satisfies equation (1), and
2. Each of $y^{[n]}, y^{[n-1]}, \cdots, y^{[1]}$ can be eliminated between the equations

$$\begin{cases} F(y, y^{[i]}) = P_n(x) \\ F^{[1]}(y, y^{[i]}) = P_n^{[1]}(x) \\ \quad \cdots \\ F^{[n]}(y, y^{[i]}) = P_n^{[n]}(x) = c, \quad \text{a constant} . \end{cases} \tag{2}$$

Note: Equations (2) in the definition exist because the first $n$ derivatives with respect to $x$ of equation (1) exist.

**2. Existence of a unique polynomial solution.** *2.1 Lemma for the homogeneous case*. To obtain uniqueness, it is necessary to prove the following lemma

*LEMMA: For each positive integer $i$, let $j_i$ denote a non-negative integer such that $j_i < i$. Then there does not exist a polynomial solution to the differential equation*

$$y + \sum_{i=1}^{i=k} S_{j_i}(x) y^{[i]} = 0 . \tag{3}$$

In equation (3), $S_{j_i}(x)$ is a polynomial of degree less than $i$, or it is zero. Hence the coefficient of each derivative of $y$ is either constant, or a non-constant polynomial of degree less than the order of the derivative of $y$ in that term. An example of equation (3) is

$$y + (ax^3 + bx)y^{[5]} = 0 . \tag{3.1}$$

This lemma may be easily proved by an indirect argument; i.e. suppose there is some polynomial (of any degree) which does satisfy the differential equation (3). Such a polynomial must satisfy the equation for all $x$, so that the coefficient of the term with the greatest exponent must be zero, which is contrary to our definition of a polynomial. This contradiction proves the lemma.

*2.2 Theorem on the unique polynomial solution of the non-homogeneous case*. Based on the preceding lemma, the following theorem may be stated.

*THEOREM: For each positive integer $i$ let $j_i$ denote a non-negative integer such that $j_i < i$. Then there exists one and only one polynomial $R_n(x)$ such that $y = R_n(x)$ satisfies the differential equation:*

$$y + \sum_{i=1}^{i=k} S_{j_i}(x) y^{[i]} = P_n(x) . \tag{4}$$

Note: An example of equation (4) is

$$y + ay^{[1]} + bxy^{[2]} + cx^5y^{[6]} = P_9(x) . \tag{4.1}$$

*PROOF:* The existence of a polynomial solution to equation (4) will be shown by induction, and the uniqueness will be shown by the use of the lemma in section 2.1. To prove the theorem of this section the following induction statement will be proved.

If $m$ is a positive integer, such that $y = R_{m-1}(x)$ satisfies

(4.2) $$y + \sum_{i=1}^{i=k} S_{j_i}(x) y^{[i]} = P_{m-1}(x)$$

then there exists a polynomial $T_m(x)$ such that $y = T_m(x)$ satisfies

(4.3) $$y + \sum_{i=1}^{i=k} S_{j_i}(x) y^{[i]} = P_m(x) .$$

The hypothesis of the above statement is satisfied for $m = 1$, because then $m-1 = 0$, and the polynomial $y = R_0(x) = P_0(x)$ satisfies equation (4.2). Differentiate equation (4.3), and obtain

(4.4) $$y^{[1]} + \sum_{i=1}^{k} S_{j_i}(x) y^{[i+1]} + \sum_{i=2}^{k} S_{j_i}^{[1]}(x) y^{[i]} = P_m^{[1]}(x) ,$$

which can be written as

(4.5) $$y^{[1]} + \sum_{i=1}^{k} S_{j_i}(x) y^{[i+1]} + \sum_{i=1}^{k} S_{j_{i+1}}^{[1]}(x) y^{[i+1]} = P_m^{[1]}(x) ;$$

where

$$S_{j_{k+1}}(x) \equiv 1 , \quad \text{and} \quad S_{j_{k+1}}^{[1]}(x) \equiv 0 .$$

Since $S_{j_{i+1}}(x)$ is a polynomial of degree less than $i+1$, then $S_{j_{i+1}}^{[1]}(x) = K_{p_i}(x)$ where $K_{p_i}(x)$ is a polynomial of degree less than $i$. Let

$$y^{[1]} = Y , \quad y^{[i+1]} = Y^{[i]} , \quad i = 1, 2, \dots, k \quad \text{and} \quad P_m^{[1]}(x) = Q_{m-1}(x) .$$

Rewrite equation (4.5) as,

(4.6) $$Y + \sum_{i=1}^{k} S_{j_i}(x) Y^{[i]} + \sum_{i=1}^{k} K_{p_i}(x) Y^{[i]} = Q_{m-1}(x) .$$

Let $S_{j_i}(x) + K_{p_i}(x) = A_{p_i}(x)$ which is a polynomial of degree less than $i$. Write equation (4.6) as

(4.7) $$Y + \sum_{i=1}^{k} A_{p_i}(x) Y^{[i]} = Q_{m-1}(x) .$$

By the hypothesis of the induction statement, equation (4.7) has a polynomial solution of degree $m-1$, i. e. $Y = B_{m-1}(x)$ satisfies equation

(4.7).

$$Y^{[i-1]} = B_{m-1}^{[i-1]}(x) = y^{[i]} .$$

Upon substituting for $y^{[i]}$ in equation (4.3) we get

$$y + \sum_{i=1}^{k} S_{j_i}(x) B_{m-1}^{[i-1]}(x) = P_m(x) ,$$

and solving for $y$, we have

$$y = P_m(x) - \sum_{i=1}^{k} S_{j_i}(x) B_{m-1}^{[i-1]}(x) = T_m(x)$$

$T_m(x)$ is a polynomial solution of equation (4.3); hence for any non-negative integer $n$, there exists a polynomial solution to equation (4). It will now be shown that there is not more than one polynomial solution to equation (4). Suppose there are two distinct polynomials $R_n(x)$ and $Q_m(x)$ such that each satisfies equation (4), then

$$(4.8) \qquad R_n(x) - Q_m(x) + \sum_{i=1}^{k} S_{j_i}(x)(R_n^{[i]}(x) - Q_m^{[i]}(x)) = 0 .$$

Since each of $R_n(x)$ and $Q_m(x)$ is a polynomial and they are different, then their difference is a polynomial which satisfies equation (4.8), but this contradicts the lemma of section 2.1. Therefore there do not exist two distinct polynomials which satisfy equation (4). Hence the theorem of section 2.2 is proved.

*2.3 COROLLARY : If $m$ is a positive integer, $m \leqq n$, and for each positive integer $i$, $j_i$ denotes a non-negative integer less than $i$, then the differential equation*

$$y^{[m]} + \sum_{i=1}^{k} S_{j_i}(x) y^{[m+i]} = P_n(x) ,$$

*has an m-parameter family of polynomial solutions.*

*PROOF :* Let $Y = y^{[m]}$, then the equation becomes

$$Y + \sum_{i=1}^{i=k} S_{j_i}(x) Y^{[i]} = P_n(x) ,$$

which has a unique polynomial solution for $Y$. Since $Y = R_n(x) = y^{[m]}$, we can take the anti derivative $m$ times and get $y$ as an $m$-parameter family of polynomial solutions.

## 3. Application of differentiation to obtain the polynomial solution.

*THEOREM. The polynomial solution of equation* (4) *can be obtained by the method of differentiation.*

*PROOF :* The proof of this theorem requires only that the two conditions of the definition in section 1.3 are satisfied by equation (4); i. e.

1. Equation (4) has a polynomial solution of degree $n$. This is true by the theorem of section 2.2.

2. Each of $y^{[n]}$, $y^{[n-1]}$, ..., $y^{[1]}$, can be eliminated between the equations

$$y+\sum_{i=1}^{k} S_{j_i}(x)y^{[i]} = P_n(x) \tag{4}$$

$$\begin{cases} (y+\sum_{i=1}^{k} S_{j_i}(x)y^{[i]})^{[1]} = P_n^{[1]}(x) \\ \quad \cdots \\ (y+\sum_{i=1}^{k} S_{j_i}(x)y^{[i]})^{[n]} = P_n^{[n]}(x) \end{cases} \tag{5}$$

Differentiating equation (4) produces the equations of (5), which is a system of $n$ equations with $n$ unknowns (consider each of $y^{[n]}$, $y^{[n-1]}$, ..., $y^{[1]}$, as an unknown.) In the system (5), $P_n^{[n]}(x)$ is a non-zero constant, hence a polynomial of degree zero. The derivative of order $m$ of any term $S_{j_i}(x)y^{[i]}$ of equation (4) contains no derivatives of $y$ of order $m$ or lower. Thus in each equation of (5), the lowest order derivative always comes from the first term. This permits us to solve for the lowest order derivative of $y$ in terms of higher derivatives. Starting with the left member of the last equation in (5) and solving for $y^{[n]} = P_n^{[n]}(x)$, *and making successive substitutions in the preceding equations,* we find $y^{[n-1]}$, $y^{[n-2]}$, ..., $y$ respectively as polynomials of degrees 1, 2, ⋯, $n$. Thus $y$ is obtained as a polynomial of degree $n$, as required.

**4. Discussion.** In finding the general solution to a linear non-homogeneous differential equation, it is sometimes convenient to find a particular solution to the equation as it stands, i. e. to the "complete equation." The general solution is then obtained by adding such a particular solution to the general solution of the homogeneous or "reduced" equation. If the equation is one of the types discussed here, then a particular solution is a polynomial which can be obtained by the use of differentiation. Two other methods used for this purpose are: the *method of undetermined coefficients*, and simply by *inspection.*

In some cases the use of differentiation is more productive. Differentiation may be applied to other types of differential equations also. For example, there are some non-linear differential equations which may, by a suitable change of variable, be changed into linear forms which can be solved by differentiation.

Obtaining a polynomial solution to a differential equation is more important than one might, at first, suspect. If a linear differential equation is of the type $F(y, y^{[i]}) = f(x)$, where $f$ is a non-polynomial analytic function, then a polynomial may be substituted for the actual function, and the pattern of the resulting polynomial solution could lead to a "guess" at a particular solution of the given differential equation. To illustrate the preceding statement (which we will call an "algorithm,") we consider the following example.

*Problem:* Find a particular solution to:

$$y + y^{[1]} = e^{x} . \tag{6}$$

*Solution:* By the above algorithm we substitute

$$y + y^{[1]} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

and obtain the 4th degree polynomial solution

$$R_4(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} .$$

Noting the pattern of $R_4(x)$ and recalling that

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

we guess $y = \cosh x$ as a solution of equation (6), which is actually the case.

# DYNAMIC PROGRAMMING AND "DIFFICULT CROSSING" PUZZLES

RICHARD BELLMAN, The Rand Corporation

**1. Introduction.** In a recent paper in this journal [1], B. Schwartz used a method based upon graph theory and topological considerations to study a familiar type of puzzle, classified in books on mathematical recreations as "difficult crossings." The object of this paper is to point out that the theory of dynamic programming [2], [3], furnishes a uniform approach to questions of this nature.

**2. Example.** Typical of the poser presented in a puzzle section of a newspaper or magazine is the following:

"A group consisting of three cannibals and three missionaries seeks to cross a river. A boat is available which will hold two people, and which can be navigated by any combination of cannibals and missionaries involving one or two people. If the missionaries on either side of the river, or in the boat, are outnumbered at any time by cannibals, the cannibals will indulge in their anthropophagic tendencies and do away with the missionaries. What schedule of crossings can be devised to permit the entire group of cannibals and missionaries to cross the river safely?"

In the next section we shall formulate the problem in more general terms and then turn to the dynamic programming formulation.

**3. General Problem.** Let us now consider the more general situation in which we start with $m_1$ cannibals and $n_1$ missionaries on one side of the river and $m_2$ cannibals and $n_2$ missionaries on the other. Let the rule be that on one bank we have a constraint $R_1(m_1, n_1) \geqq 0$ to prevent the missionaries from being devoured, and a similar constraint $R_2(m_2, n_2) \geqq 0$ on the other, and a constraint $R_3(m, n) \geqq 0$ in the boat, capable of carrying at most $k$ people.

It is not at all clear when it is possible to schedule a safe crossing. Consequently, we shall begin by treating the following problem. What is the maximum number of people that can be transported from one bank, say bank one to the other, without permitting cannibalism?

**4. Dynamic Programming Formulation.** Since the total number of cannibals and of missionaries stays constant throughout the process, the state of the system at any time is specified by the numbers $m_1$ and $n_1$ defined above.

Let us then introduce the function

(1) $f_N(m_1, n_1)$ = the maximum number of people on the second bank at the end of $N$ stages, starting with $m_1$ cannibals and $n_1$ missionaries on the first bank and quantities $m_2$ and $n_2$ respectively on the second bank.

We shall suppose that it is permissible at any stage to send no people back to the first bank from the second bank if everybody is already on the second bank.

One stage of the process consists of sending $x_1$ cannibals and $y_1$ missionaries from the first bank to the second bank and then of sending $x_2$ cannibals and $y_2$ missionaries back to the first bank.

Using the principle of optimality [2], p. 83, we obtain the recurrence relation

$$f_N(m_1, n_1) = \max_{x,\ y} f_{N-1}(m_1 - x_1 + x_2,\ n_1 - y_1 + y_2)\ , \tag{2}$$

for $N \geqq 2$, where the varisbles $x_1, x_2, y_1, y_2$ are subject to the constraints

(3) (a) $$0 \leqq x_1 \leqq m_1\ , \quad 0 \leqq y_1 \leqq n_1\ ,$$

(b) $$0 \leqq x_2 \leqq m_2 + x_1\ , \quad 0 \leqq y_2 \leqq n_2 + y_1\ ,$$

(c) $$x_1 + y_1 \leqq k\ , \quad x_2 + y_2 \leqq k\ ,$$

(d) $$R_3(x_1, y_1) \geqq 0\ , \quad R_3(x_2, y_2) \geqq 0\ ,$$

(e) $$R_1(m_1 - x_1,\ n_1 - y_1) \geqq 0\ ,$$

(f) $$R_1(m_1 - x_1 + x_2,\ n_1 - y_1 + y_2) \geqq 0\ ,$$

(g) $$R_2(m_2 + x_1,\ n_2 + y_1) \geqq 0\ ,$$

(h) $$R_2(m_2 + x_1 - x_2,\ n_2 + y_1 - y_2) \geqq 0\ .$$

There are sets of $x_1, x_2, y_1, y_2$ satisfying these constraints, since by assumption $x_1 = x_2 = y_1 = y_2 = 0$ satisfies them.

For $N = 1$, we have

$$f_1(m_1, n_1) = \max_{x,\ y} [(m_2 + x_1) + (n_2 + y_1)]\ , \tag{4}$$

where $x_1, y_1$ are subject to the foregoing constraints.

**5. Discussion.** For small values of $N$ and of $m_1, n_1, m_2, n_2$ the values of $f_N(m_1, n_1)$ can readily be computed by hand. In many cases, the constraints will be such that there will be a unique scheduling which maximizes. Observe that the foregoing procedure will simultaneously determine the minimum number of total crossings necessary for the transference of all the people from one bank to the other whenever it is possible. To obtain this number, we merely continue the process until we find a value for $N$ for which

$$f_N = m_1 + n_1 + m_2 + n_2\ .$$

In general, the solution to the crossing problem can be obtained in a matter of a few seconds using a digital computer; cf. [3].

Let us note finally that similar problems involving the pouring of water or wine from one type of jug to another to obtain a mixture of the desired type and volume are treated by dynamic programming techniques in [2], p. 99, problem 38.

*REFERENCES*

1. B. Schwartz, *An analytic method for the "difficult crossing" puzzles,* Mathematics Magazine, vol. 34, number 3, Jan.-Feb., 1961, pp. 187-193.

2. R. Bellman, *Dynamic Programming,* Princeton, New Jersey, 1957.

3. R. Bellman and S. Dreyfus, *Computational Aspects of Dynamic Programming,* Princeton, New Jersey, 1962.

---

## TO A POINT

*Wert thou expanded millionfold*
*Thy scope would still be not*
*The millionthfold contraction of*
*A microscopic dot.*
*Thou'rt nothing more or nothing less*
*Than concentrated nothingness –*
*And yet we name thee, give thee weights,*
*And clothe thee in coordinates.*

*But yet again, thy aggregates*
*Form segments, which in turn form plates*
*Which coalesce to one thing more –*
*Space, nothing less and nothing more.*
*And space, contracted millionfold,*
*Would still suffice to keep*
*The millionthfold expansion of*
*The macrocosmic sweep.*

Marlow Sholander

---

# AN EXTENSION OF THE KURATOWSKI CLOSURE AND COMPLEMENTATION PROBLEM

THOMAS A. CHAPMAN, West Virginia University

**1. Introduction.** Kuratowski [2] has found that if $A$ is a subset of a topological space, then at most 14 different sets can be constructed from $A$ by successive application of the closure ( $^{-}$ ) operator and the complementation ( $'$ ) operator. Since then these results have been generalized by the study of the notion of a closure algebra [1].

This theorem seems to be an end in itself, that is without any further application. Indeed, it may even be considered by some to be trivial. If, however, we consider generalizations of the fact that from a given set $n$ distinct sets may be constructed by closure and complementation, then the problem leads to some apparently new and interesting results. We shall show here that the Kuratowski problem leads to a characterization of topological spaces in terms of abstractions of the operators to general sets.

**2. First and second line functions.** For the purposes of this paper we will need the following definition.

2.1. *Definition. If $A$ is a subset of a topological space, then $A^*$ denotes the set consisting of the elements $B_1^A$, $B_2^A$, $B_3^A$, $\cdots$ where $B_1^A = A$, $B_2^A = A'$, $B_3^A = A'^{-}$, $B_4^A = A'^{-\prime}$, etc.*

We remark that one half of the Kuratowski problem may be expressed as follows:

2.2. *If $A$ is a subset of a topological space then the number of distinct elements in $A^*$ does not exceed 8.*

For the sake of completeness we recall briefly a proof of this suggested to the writer by Mr. Richard L. Stalnaker.

If $A^\circ$ denotes the interior of $A$, then it is readily seen that $B_9^A = A'^{-\prime-\prime-\prime-} = A^{\circ-\circ-}$ and $B_5^A = A^{\circ-}$. Since $A^{\circ-\circ} \subset A^{\circ-}$ we have $A^{\circ-\circ-} \subset A^{\circ-}$. From $A^\circ \subset A^{\circ-}$ we have $A^\circ \subset A^{\circ-\circ}$ and $A^{\circ-} \subset A^{\circ-\circ-}$. Therefore $B_9^A = B_5^A$ and for $i \geqq 9$, $B_i^A = B_j^A$ $(5 \leqq j \leqq 8)$.

2.3. *Definition. Let $A$ be a subset of a topological space $S$. Relative to* 2.1 *let $h$ be the first integer such that $B_{h+1}^A = B_i^A$ for some $i$ $(1 \leqq i \leqq h)$. Define mappings $k_1$ and $k_2$ of the family of all subsets of $S$ into $P$ (the set of all natural numbers) such that $k_1(A) = h$ and $k_2(A) = i$. The numbers $k_1(A)$ and $k_2(A)$ will be called the first and second line values of $A$ and the functions $k_1$ and $k_2$ will be called the first and second line functions for the space $S$.*

In the remainder of this paper we shall assume the topological spaces considered to be non-empty.

**3. Properties of line functions.** It is instructive to find what numerical values the line functions may assume in an arbitrary space.

3.1. *Theorem. Let* $A$ *be a subset of a topological space. Then*

(i) $k_1(A) = 2, 4, 6$, or $8$;

(ii) $k_1(A) = 2$ implies that $k_2(A) = 2$;

(iii) $k_1(A) = 4$ implies that $k_2(A) = 4$ or $1$;

(iv) $k_1(A) = 6$ implies that $k_2(A) = 6$ or $3$;

(v) $k_1(A) = 8$ implies that $k_2(A) = 5$.

*Proof.* (i) From 2.2 we have $k_1(A) = h$ $(1 \leqq h \leqq 8)$. Since $B_1^A$ and $B_2^A$ are distinct we have $k_1(A) \geq 2$. If $k_1(A) = h$ (odd) then we must have $B_{h+1}^A = B_i^A$ $(i < h)$. But this implies that $B_h^A = B_j^A$ $(j < h)$, which contradicts the fact that $k_1(A) = h$.

(ii) Either $B_3^A = B_1^A$ or $B_3^A = B_2^A$. The former equation leads to a contradiction since the space is assumed non-empty.

(iii) If $B_5^A = B_3^A$ then $A'^{-\prime-} = A'^{-}$ which cannot occur in a non-empty space. If $B_5^A = B_2^A$ then $A'^{-\prime-} = A'$ which cannot occur in a non-empty space.

(iv) If $B_7^A = B_5^A$ then $A'^{-\prime-\prime-} = A'^{-\prime-}$ which cannot occur in a non-empty space. By the same token $B_7^A = B_4^A$ cannot occur. If $B_7^A = B_2^A$ then $B_3^A = B_2^A$ since $B_2^A$ is closed and $B_3^A$ is the closure of $B_2^A$. Finally if $B_7^A = B_1^A$ then from the reasoning used in 2.2 we have $B_1^A = B_8^A$ which is a contradiction.

(v) From 2.2 we have $B_9^A = B_5^A$.

The following theorem exhibits the close relationship between line functions and open sets.

3.2. *Theorem. A subset* $A$ *of a topological space is open if and only if* $k_1(A) = 2$.

*Proof.* If $A$ is open then $A'^{-} = A'$ and $k_1(A) = 2$. If $k_1(A) = 2$ then $A'^{-} = A'$ and $A$ is open.

The next definition is a natural outgrowth of the other half of the Kuratowski problem.

3.3. *Definition. Let* $A$ *be a subset of a topological space. Denote by* $A^{**}$ *the set consisting of the elements* $C_1^A, C_2^A, C_3^A, \ldots$ *where* $C_1^A = A'$, $C_2^A = A$, $C_3^A = A^-$, $C_4^A = A^{-\prime}$, $C_5^A = A^{-\prime-}$, *etc.*

We observe that the other part of Kuratowski's problem may now be expressed as follows:

3.4. *If* $A$ *is a subset of a topological space then the number of distinct elements in* $A^{**}$ *does not exceed 8.*

The proof follows readily from 2.2 since $A^{**} = (A')^*$ and the number of distinct elements in $(A')^*$ does not exceed 8.

We are now able to give a number of interesting small theorems.

3.5. *Theorem. Let $A$ be a subset of a topological space. Then $k_1(A')$ is the first integer such that $C^A_{m+1} = C^A_i$ for some $i \leqq m = k_1(A')$ and $k_2(A') = i$.*

*Proof.* By definition $k_1(A')$ is the first integer such that $B^{A'}_{m+1} = B^{A'}_i$ for some $i \leqq m = k_1(A')$ and $k_2(A') = i$. The theorem then follows since $B^{A'}_j = C^A_j$.

3.6. *Theorem. A subset $A$ of a topological space is closed if and only if $k_1(A') = 2$.*

*Proof.* This is a direct consequence of 3.2.

3.7. *Theorem. Let $A$ be a subset of a topological space. If $k_1(A) = 2$, then $k_1(A') \neq 8$.*

*Proof.* If $k_1(A) = 2$ then $A^-$ is the closure of an open set. This implies that $A^{-\prime-\prime-} = A^-$ and it follows that $k_1(A') \neq 8$.

3.8. *Theorem. Let $A$ be a subset of a topological space. Then $k_1(A^-) = k_1(A') - 2$ and $k_2(A^-) = k_2(A') - 2$ if and only if $k_1(A') \neq 2$ and $k_2(A') \neq 1$.*

*Proof.* Suppose $k_1(A') \neq 2$ and $k_2(A') \neq 1$. This implies $k_2(A') \neq 2$. Now $B^{A^-}_1 = C^A_3$, $B^{A^-}_2 = C^A_4$, .... Therefore $k_1(A^-) = k_1(A') - 2$ and $k_2(A^-) = k_2(A') - 2$. Conversely suppose $k_1(A^-) = k_1(A') - 2$ and $k_2(A^-) = k_2(A') - 2$. If $k_1(A') = 2$ then $k_1(A^-) = 0$ which is a contradiction. If $k_2(A') = 1$ then $k_2(A^-) = -1$ which is a contradiction.

3.9. *Corollary. Let $A$ be a subset of a topological space. Then $k_1(A^{\circ\prime}) = k_1(A) - 2$ and $k_2(A^{\circ\prime}) = k_2(A) - 2$ if and only if $k_1(A) \neq 2$ and $k_2(A) \neq 1$.*

*Proof.* This follows readily by substituting $A'$ in place of $A$ in 3.8.

3.10. *Definition. Let $S$ be a topological space. Define a mapping $K$ of the family of all subsets of $S$ into $P \times P$ ($P$ = set of all natural numbers) such that if $A \subset S$, then $K(A) = (k_1(A), k_2(A))$. This function $K$ will be called the generalized line function for the space $S$.*

We remark that from 3.1 it is obvious that the only values which $K(A)$ may assume are included among the following elements of $P \times P$: (2, 2), (4, 1), (4, 4), (6, 3), (6, 6), (8, 5). It is not difficult to find examples of spaces such that given some subset $A$, then the image of $A$ under $K$ is any one of these elements.

In view of the apparent close relationship between the first and second line functions and the closure and interior operators it seems plausible that a topology for a set may be defined in terms of abstractions of these functions. A solution to this problem is taken up in the next section of this paper. Before proceeding to this we make the notational definition

3.11. *Definition. The subset of* $P \times P$ *consisting of the elements* (2, 2), (4, 1), (4, 4), (6, 3), (6, 6), (8, 5) *will be denoted by* $I$.

**4. Set line functions.** In 2.3 the line functions were defined for a topological space. We now define line functions for an arbitrary set.

4.1. *Definition. Let* $S$ *be any set. Define functions* $k_1'$ *and* $k_2'$ *on the family of all subsets of* $S$ *into* $P$ *such that given any* $A \subset S$, *then the ordered pair* $(k_1'(A), k_2'(A))$ *is an element of* $I$. *These functions will be called the first and second line functions for the set* $S$.

4.2. *Definition. It will be convenient to define an operator* $^\propto$ *on the family of all subsets of* $S$ *into the family of all subsets of* $S$ *such that*

$$A^\propto = \cup\{B : B \subset A,\ k_1(B) = 2\}\ .$$

For the proof of the main result of this paper we will need the following simple lemma.

4.3. *Lemma. Let* $A$ *be a subset of a topological space* $S$ *with second line function* $k_2$. *Then* $k_2(A) = 1$ *if and only if* $A^{\circ\prime\circ\prime} = A$ *and* $A^{\circ\prime\circ\prime} \neq A^\circ$.

*Proof.* Suppose $k_2(A) = 1$. Then $A'^- \neq A'$, $A'^{-\prime-} \neq A'^{-\prime}$ and $A'^{-\prime-} = A$. But since $B'^{-\prime} = B^\circ$ for any $B \subset S$ we have $A^\circ \neq A$, $A^{\circ\prime\circ\prime} \neq A^\circ$ and $A^{\circ\prime\circ\prime} = A$. Now suppose on the other hand that $A^{\circ\prime\circ\prime} = A$ and $A^{\circ\prime\circ\prime} \neq A^\circ$. Then by definition this implies that $k_2(A) = 1$.

We now state and prove our main result.

4.4. *Theorem. Let* $S$ *be a set and* $k_1'$, $k_2'$ *the first and second line functions for* $S$ *and suppose the following conditions are true :*

(i) $k_1'(S) = 2$ *and* $k_1'(A^\propto) = 2$ *for any* $A \subset S$;

(ii) *If* $k_1'(A) = 2$ *and* $k_1'(B) = 2$, *then* $k_1'(A \cap B) = 2$;

(iii) $k_1'(A^{\propto\prime}) = k_1'(A) - 2$ *and* $k_2'(A^{\propto\prime}) = k_2'(A) - 2$ *if and only if* $k_1'(A) \neq 2$ *and* $k_2'(A) \neq 1$;

(iv) $k_2'(A) = 1$ *if and only if* $A^{\propto\prime\propto\prime} = A$ *and* $A^{\propto\prime\propto\prime} \neq A^\propto$.

*Then the family of all subsets* $A$ *of* $S$ *for which* $k_1'(A) = 2$ *forms a topology for* $S$, *and* $^\propto$ *is the interior operator relative to this topology. Also the first and second line functions for the space* $S$ *are the first and second line functions* $k_1'$, $k_2'$ *for the set* $S$.

*Proof.* Let $t$ denote the family of all subsets $A$ of $S$ such that $k_1'(A) = 2$. It will be shown that $t$ is a topology for $S$. Suppose $A$ and $B$ belong to $t$. Then from (ii) it follows that $A \cap B \in t$. Next let $s$ be a subset of $t$ and let $D = \cup\{A : A \in s\}$. We must show that $k_1'(D) = 2$. This will easily follow from (i) if it can be shown that $D^\propto = D$. The inclusion $D^\propto \subset D$ is obvious. To prove the reverse inclusion, let $p \in D$. Then there exists an $A$ such that $p \in A \subset D$ and $k_1'(A) = 2$. Thus $p \in D^\propto$ and equality follows. Therefore $D \in t$. From (i) it follows that $\emptyset$ and $S$ are members of $t$. We may now conclude that $t$ is a topology for $S$. It is also quite obvious that the operator $\propto$ is actually the interior operator for the space $S$. It now remains to prove that $k_1'$ and $k_2'$ are the first and second line functions for the space $S$.

Let $k_1$ and $k_2$ be the first and second line functions for the space. Since $A$ is open if and only if $k_1(A) = k_1'(A) = 2$, then $k_1(A) = 2$ if and only if $k_1'(A) = 2$ and $k_2(A) = 2$ if and only if $k_2'(A) = 2$. Since $\propto$ is the interior operator, (iv) and 4.3 together imply $k_1(A) = 4$ and $k_2(A) = 1$ if and only if $k_1'(A) = 4$ and $k_2'(A) = 1$. The remainder of the proof follows from simple reductions. For example, $k_1(A) = 4$ and $k_2(A) = 4$ imply $k_1(A^{\propto\prime}) = 2$ and $k_2(A^{\propto\prime}) = 2$. But from previous considerations we have $k_1'(A^{\propto\prime}) = 2$ and $k_2'(A^{\propto\prime}) = 2$. From (iii) we have $k_1'(A) = k_1'(A^{\propto\prime}) + 2 = 4$, and $k_2'(A) = k_2'(A^{\propto\prime}) + 2 = 4$. After a finite number of reductions the theorem follows.

## *REFERENCES*

1. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. Vol. 25, rev. ed., 1948, page 180.

2. C. Kuratowski, *L'opération $\bar{A}$ de l'analysis situs*, Fundamenta Mathematicae, vol. 3, 1922, pp. 182-199.

# NINETEEN HUNDRED AND SIXTY-TWO

H. W. GOULD, West Virginia University

The new year of 1962 has many fascinating number properties already as may be seen from the following amazing relationships :

$$666 + 6^4 = 1962$$

$$1^3 + 9^3 + 6^3 + 2^3 = 53(1 + 9 + 6 + 2)$$

$$1^3 + 9^3 + 6^3 + 2^3 = 7(1^2 + 9^2 + 6^2 + 2^2) + 100$$

$$1^4 + 9^4 + 6^4 + 2^4 = 3^8 + 1313$$

$$1! + 9! + 6! + 2! = 603^2 - 6$$

$$2(1 \cdot 9 \cdot 6 \cdot 2) = 6^3$$

$$1962^2 = 157^3 - 143^2$$

$$(1 + 9 + 6 + 2)(1^2 + 9^2 + 6^2 + 2^2) = 13^3 - 1$$

$$(19 + 62) - (1 + 9)(6 + 2) = 1$$

$$(1 - 9 + 6 + 2)(1^{13} + 9^{13} + 6^{13} + 2^{13}) = 0$$

$$1^2 \cdot 9^2 + 6^2 \cdot 2^2 = (6 + 9)^2$$

$$-1^4 + 9^4 - 6^4 + 2^4 = \text{one mile} = 5280$$

$$\begin{vmatrix} 1 & 9 & 6 & 2 \\ 9 & 6 & 2 & 1 \\ 6 & 2 & 1 & 9 \\ 2 & 1 & 9 & 6 \end{vmatrix} = 4(11^3 + 1) , \quad \begin{vmatrix} 1 & 9 & 6 & 2 \\ 9 & 1 & 2 & 6 \\ 6 & 2 & 1 & 9 \\ 2 & 6 & 9 & 1 \end{vmatrix} = 12^3$$

Finally, we see that

$$1 \cdot 9 + 9 \cdot 6 + 6 \cdot 2 + 2 \cdot 1 = 73 ,$$

and the number 73 in amateur radio circles means "best regards".

# AN APPLICATION OF GENERATING SERIES

LEO MOSER, University of Alberta

Although the notion of generating series is a very important one in number theory, combinatorial analysis, probability theory and other branches of mathematics, there are perhaps not enough well-known examples which illustrate the applicability of this notion in different situations. In this note we present what is probably a new example of this type. The results we derive could be obtained by purely elementary methods as well, but this does not detract from the elegance of the generating series approach.

Let us seek a sequence of non-negative integers $A = \{a_1 < a_2 < \cdots\}$ such that every non-negative integer $n$ can be represented *uniquely* in the form $n = a_i + 2a_j$. To this end we define a function $f(x)$ by

$$f(x) = \sum_{i=1}^{\infty} x^{a_i}, \quad |x| < 1 . \tag{1}$$

Now the number of representations of each non-negative integer $n$ in the form $a_i + 2a_j$ will be the coefficient of $x^n$ in the expansion of $f(x)f(x^2)$. Thus, since each integer is to have a unique representation we must have

$$f(x)f(x^2) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} . \tag{2}$$

From (2) we obtain

$$\frac{f(x)f(x^2)}{f(x^2)f(x^4)} = \frac{1-x^2}{1-x} \tag{3}$$

or

$$f(x) = (1+x)f(x^4) . \tag{4}$$

Iteration of (4) and taking into account the fact that $f(x^n) \to 1$ as $n \to \infty$ yields

$$f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64}) \cdots . \tag{5}$$

The coefficient of $x^n$ on the right hand side of (5) is the number of representations of $n$ as the sum of distinct powers of 4. Since every integer has a unique representation as a sum of distinct powers of 2, a number will have at most one representation as a sum of distinct powers of 4 and our argument shows that the set $A$ must consist precisely of those numbers which have such a representation. The argument actually shows that if a set $A$ exists and has the required properties then it must be the unique set described above, but having found the set it is easily seen that it does indeed have the required properties. The required set begins with

$$A : \{0, 1, 4, 5, 16, 17, 20, 21, 64, \cdots\} .$$

A by-product of this argument is still another example of an explicit (1-1) correspondence between the non-negative integers $n$ and the positive pairs of integers $(i, j)$. This is an immediate consequence of the unique solvability of the equation $n = a_i + 2a_j$, i. e. given $n$ the $i$ and $j$ are uniquely determined.

As a further application of the method one can prove, in similar fashion, that for every integer $k > 1$ there is a unique set $A_k$ of non-negative integers such that every integer $n$ can be uniquely expressed in the form $n = a + bk$, with $a$ and $b$ elements of $A_k$. On the other hand such a set cannot exist for $k = 1$ for then with $(x)$ defined as in (1),

$$f^2(x) = \frac{1}{1-x} \tag{6}$$

so that

$$f(x) = (1-x)^{-1/2} \tag{7}$$

which does not have an expansion of the form (1).

---

## RATIONAL APPROXIMATIONS OF $e$

CHARLES W. TRIGG, Los Angeles City College

It is well-known that (2721)/(1001) $\doteq$ 2.7182817 approximates the value of $e$, being accurate to 6 decimal places. This is equivalent to

$$e \doteq \frac{4}{7}+\frac{16}{11}+\frac{9}{13} = \frac{11}{7}+\frac{5}{11}+\frac{9}{13} = \frac{4}{7}+\frac{5}{11}+\frac{22}{13}.$$

In all three cases, the denominators are consecutive primes. In the first sum, the numerators are consecutive squares. In the second sum, the numerators are all odd and their sum is the next consecutive square. In the third sum, the sum of the numerators equals the sum of the denominators, i. e. 31.

The approximating fraction may also be written as

$$\frac{877+907+937}{7 \cdot 11 \cdot 13}$$

in which the denominator is the product of three consecutive primes and the numerator is the sum of three primes in arithmetic progression. The numerator may be written as the sum of three primes in A. P. in 25 other ways, in all of which 907 is the mean. The smaller terms of the A. P.'s

*(Continued on page 54.)*

$$BR^{-1}RR^{-1}B'(AW^{-1}A')^{-1} \quad \text{or} \quad BR^{-1}B'(AW^{-1}A')^{-1}$$

leaving $I - BR^{-1}B'(AW^{-1}A')^{-1}$; showing that $H$ is idempotent, i. e., $H = H \cdot H = H^n$.

$I - H$ is also idempotent; thus

$$(I-H)(I-H) = I - H - H + H \cdot H = I - H - H + H = I - H .$$

If one considers the model

$$2v_1 - v_2 + 3v_3 + v_4 + 4u_1 - u_2 + 1.8 = 0$$

$$v_1 + 3v_2 + v_3 - 2v_4 + u_1 + 6u_2 - 3.6 = 0$$

$$5v_1 + 2v_2 - v_3 + 8v_4 - 2u_1 - 3u_2 + 6.4 = 0$$

with

$$W = \begin{bmatrix} 1 & & & \\ & .3333 & & \\ & & .5 & \\ & & & .5 \end{bmatrix}$$

then it turns out that

$$H = \begin{bmatrix} .0422 & .0655 & .1165 \\ .0168 & .0270 & .0485 \\ .3354 & .5213 & .9309 \end{bmatrix} .$$

It is easily shown that $H$ reproduces itself and that the determinant value of $H$ is zero.

---

*(Continued from page 38.)*

are: 3, 13, 31, 37, 61, 67, 73, 151, 157, 193, 271, 283, 331, 367, 433, 487, 523, 577, 601, 613, 643, 661, 727, 751, and 823.

Of the fractions with denominators less than 100, the one most closely approximating $e$ by defect is (106)/(39) ≐ 2.717949, being accurate to three decimal places. The fraction most closely approximating $e$ by excess is (193)/(71) ≐ 2.718310, which is accurate to four decimal places.

---

# TEACHING OF MATHEMATICS

Edited by ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.*

## THE DISTANCE FORMULA AND CONVENTIONS FOR SIGN

THOMAS E. MOTT, The Pennsylvania State University

As a mathematician, I have always felt secure in the belief that our's is a very orderly and consistent science. Yet as a teacher faced with the task of converting others to my belief, I have often to be extremely careful lest some inconsistency creep into my own lectures. One of these occasions has been concerned with the distance formula in Analytic Geometry. The troublesome point being the convention to be adopted for the sign of this distance. Therefore I propose here to give an explanation of three such conventions, two of which are common in most texts on the subject.

Since there are a number of satisfactory proofs of the distance formula in Analytic Geometry, we shall merely be concerned here with the result itself. However, a proof which does not require the normal form, such as is to be found in "Analytic Geometry" by John W. Cell, would seem the most suitable; for I do not require the normal form in this paper. If $ax+by+c=0$ is the equation of a line then

$$d=\frac{ax_1+by_1+c}{\pm\sqrt{a^2+b^2}}$$

is the distance from this line to the point $p(x_1, y_1)$ in the plane. Notice that we have yet to adopt a convention for the sign $\pm$, and that we speak of distance from the line to the point. We shall at first be concerned only with oblique lines, hence $a\cdot b\neq 0$.

The first convention under consideration is that the sign of the radical be the same as the sign of $b$. This will then provide a distance $d$ which is positive when $p$ is "above" the line and negative when $p$ is "below" the line. But what is meant here by "above" and "below" the line? Perhaps it would be better to say "above in the $y$ sense" and "below in the $y$ sense." For by $p$ "above the line in the $y$ sense," we mean that the vertical projection of $p$ on the line is below $p$, while $p$ "below the line in the $y$ sense" means that the vertical projection of $p$ on the line is above $p$. The proof of this proposition is as follows:

The line $ax+by+c=0$ divides the plane into two half planes, the half plane "above" the line and the half plane "below" the line. These half planes contain respectively all the points "above the line in the $y$

sense" and all the points "below the line in the $y$ sense." Let $ax+by+c_1=0$ be the equation of the line thru $p(x_1, y_1)$ and parallel to the given line $ax+by+c = 0$, and assume that $p$ is "above the line in the $y$ sense." Since the line $ax+by+c_1 = 0$ is "above" the line $ax+by+c \doteq 0$, then considering the $y$ intercepts of these lines, we obtain

$$-\frac{c_1}{b} > -\frac{c}{b}.$$

If $b > 0$ we have $c_1 < c$ and if $b < 0$ we have $c_1 > c$, hence

$$0 = ax_1+by_1+c_1 < ax_1+by_1+c \qquad \text{if } b > 0$$

and

$$0 = ax_1+by_1+c_1 > ax_1+by_1+c \qquad \text{if } b < 0 .$$

But in either case $(ax_1+by_1+c)/(\text{sgn } b)$ is positive. Similarly we find that $(ax_1+by_1+c)/(\text{sgn } b)$ is negative if the point $p(x_1, y_1)$ is "below the line in the $y$ sense." Therefore we now have the desired result that

$$d = \frac{ax_1+by_1+c}{(\text{sgn } b)\cdot\sqrt{a^2+b^2}}$$

is positive if $p(x_1, y_1)$ is "above" the line and negative if $p(x_1, y_1)$ is "below" the line.

By an argument alalogous to that given above we treat the case of choosing the sign of the radical the same as the sign of $a$. We obtain

$$d = \frac{ax_1+by_1+c}{(\text{sgn } a)\cdot\sqrt{a^2+b^2}}$$

positive if $p_1(x_1, y_1)$ is "above the line in the sense of $x$" and negative if $p_1(x_1, y_1)$ is "below the line in the sense of $x$."

Next we consider the convention which assigns to the radical the sign of $c$. This will provide a distance $d$ which is positive when the origin is on the same side of the line as the point $p(x_1, y_1)$ and negative when on the opposite side.

Let the points $p(x_1, y_1)$ and (0, 0) be on the same side of the line $ax+by+c = 0$ and assume that this is the side "above the line in the $y$ sense." Then on considering the origin we obtain

$$\frac{c}{(\text{sgn } b)\cdot\sqrt{a^2+b^2}} > 0$$

so that $(\text{sgn } c) = (\text{sgn } b)$. Therefore

$$\frac{ax_1+by_1+c}{(\operatorname{sgn} c)\cdot\sqrt{a^2+b^2}}=\frac{ax_1+by_1+c}{(\operatorname{sgn} b)\cdot\sqrt{a^2+b^2}}>0\,.$$

On the other hand if the points $p(x_1, y_1)$ and (0, 0) were both on the side "below the line in the $y$ sense" then $(\operatorname{sgn} c)=-(\operatorname{sgn} b)$ and

$$\frac{ax_1+by_1+c}{(\operatorname{sgn} c)\cdot\sqrt{a^2+b^2}}=\frac{-(ax_1+by_1+c)}{(\operatorname{sgn} b)\cdot\sqrt{a^2+b^2}}>0\,.$$

Thus in either case we have

$$d=\frac{ax_1+by_1+c}{(\operatorname{sgn} c)\cdot\sqrt{a^2+b^2}}>0$$

if the points $p(x_1, y_1)$ and (0, 0) are on the same side of the line. By a similar argument one readily verifies that

$$d=\frac{ax_1+by_1+c}{(\operatorname{sgn} c)\cdot\sqrt{a^2+b^2}}<0$$

if the points $p(x_1, y_1)$ and (0, 0) are on opposite sides of the line.

Finally let us consider the lines which are parallel to the $x$ or $y$ axis, that is lines $ax+by+c=0$ for which $a$ or $b$ is zero. Let us consider first the line $by+c=0$ which is parallel to the $x$ axis. The only concept for "above" and "below" which is now admissible being in the sense of $y$. Whether $p(x_1, y_1)$ is above or below the line the distance with correct sign is $y_1-y=y_1+c/b$. From the distance formula we also obtain

$$d=\frac{by_1+c}{(\operatorname{sgn} b)\cdot\sqrt{b^2}}=\frac{by_1+c}{b}=y_1+\frac{c}{b}\,.$$

Thus the distance formula is valid for lines parallel to the $x$ axis, and by a similar argument one sees that the distance formula is valid for lines parallel to the $y$ axis. It is important to notice however, that the distance formula used with lines parallel to the $x$ axis must be the one with $(\operatorname{sgn} b)$ in the denominator, and that used for lines parallel to the $y$ axis must be the one with $(\operatorname{sgn} a)$ in the denominator.

Since the formula involving $(\operatorname{sgn} b)$ is valid for lines parallel to the $x$ axis then we derive the formula involving $(\operatorname{sgn} c)$ for lines parallel to the $x$ axis just as we did above for oblique lines. But only the formula involving $(\operatorname{sgn} a)$ is valid for lines parallel to the $y$ axis, hence to derive the formula involving $(\operatorname{sgn} c)$ for lines parallel to the $y$ axis, we merely replace the argument given above for oblique lines by one involving $(\operatorname{sgn} a)$ instead of $(\operatorname{sgn} b)$. Thus the formula

$$d=\frac{ax_1+by_1+c}{(\operatorname{sgn} c)\cdot\sqrt{a^2+b^2}}$$

is valid for all lines in the plane, and is positive if the points $p(x_1, y_1)$ and (0, 0) are on the same side of the line $ax+by+c = 0$ and negative if on opposite sides.

The proof may be more complicated, but certainly the results stated above remain valid for the distance formula in Solid Analytic Geometry. The most common form would then be that

$$d = \frac{ax_1+by_1+cz_1+d}{(\text{sgn } c)\cdot\sqrt{a^2+b^2+c^2}},$$

with $d$ then positive if the point $(x_1, y_1, z_1)$ is above the plane $ax+by+cz+d = 0$ in the $z$ sense and negative if below it.

---

## ALTERNATE PRIMES IN ARITHMETIC PROGRESSION

C. W. TRIGG, Los Angeles City College

If alternate primes are to be in arithmetic progression, the common difference must be even and $\geqq 6$. For primes > 5, the only possible sequences of the unit's digits of the primes are 1, 1, 1; 3, 3, 3; 7, 7, 7; 9, 9, 9; 7, 9, 1; 9, 1, 3; 3, 7, 1; 9, 3, 7; 1, 7, 3; 7, 3, 9; 1, 9, 7; and 3, 1, 9.

There are fifty-three triads of alternate primes < 10,000 which are in arithmetic progression. They fall into four categories in which the common differences are 6, 12, 18, and 24 with frequencies of 9, 27, 16, and 1, respectively. For example: 5, 11, 17; 139, 151, 163; 683, 701, 719; and 4373, 4397, 4421. The leading primes in each triad, grouped by common differences are:

$d = 6$: 5, 7, 11, 97, 101, 1481, 1867, 3457, 5647.

$d = 12$: 139, 167, 239, 397, 409, 479, 727, 929, 1039, 1559, 2269, 2647, 2659, 2729, 2999, 4217, 4219, 4229, 4259, 5557, 6067, 6299, 6679, 6779, 7517, 8209, 8669.

$d = 18$: 683, 691, 2423, 2731, 2801, 3001, 3371, 4093, 5153, 5261, 8293, 8563, 9203, 9221, 9643, 9661.

$d = 24$: 4373.

The smallest triad > 10,000 is 10139, 10151, 10163.

An immediate consequence is that there are six sets of four alternate primes < 10,000 in A.P., one with $d = 6$, three with $d = 12$, and two with $d = 18$. They are: 5, 11, 17, 23; 397, 409, 421, 433; 2647, 2659, 2671, 2683; 4217, 4229, 4241, 4253; 9203, 9221, 9239, 9257; and 9643, 9661, 9679, 9697.

The two sets of seven consecutive primes beginning with 5 and with 4217 both consist of one group of four and one group of three alternate primes in A.P.

# MISCELLANEOUS NOTES

Edited by ROY DUBISCH, University of Washington

Articles intended for this department should be sent to *Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.*

## ANOTHER SOLUTION OF THE CUBIC EQUATION

J. G. CAMPBELL, Albany, Kentucky

If the cubic equation $y^3+3H+G=0$ has real coefficients and $G^2+4H^3>0$, its roots may be found algebraically. If $G^2+4H^3<0$, then the trigonometric solution must be used. (See, for example, Conkwright, *Introduction to the Theory of Equations*, Ginn and Co., N. Y., 1941, pp. 75-76.) It is the purpose of this note to show that a trigonometric solution of a cubic equation is possible without a knowledge of root extraction of complex numbers by De Moivre's theorem.

*Theorem.* Given the equations

$$y^3+3Hy+G=0 \tag{1}$$

and

$$y^3+3hy+g=0 \tag{2}$$

where

$$\frac{G^2}{H^3}=\frac{g^2}{h^3}\,. \tag{3}$$

If $y_1$ is a root of (1), then $y_2=gHy_1/Gh$ is a root of (2).

*Proof.* Substituting $y_2=gHy_1/Gh$ in (2) yields $g(y_1^3+3Hy_1+G)=0$. From the trigonometric identity, $\cos^3\phi-\frac{3}{4}\cos\phi-\frac{1}{4}\cos 3\phi=0$, we see that $y=\cos\phi$ is a root of the equation

$$y^3-\tfrac{3}{4}y-\tfrac{1}{4}(\cos 3\phi)=0\,, \tag{4}$$

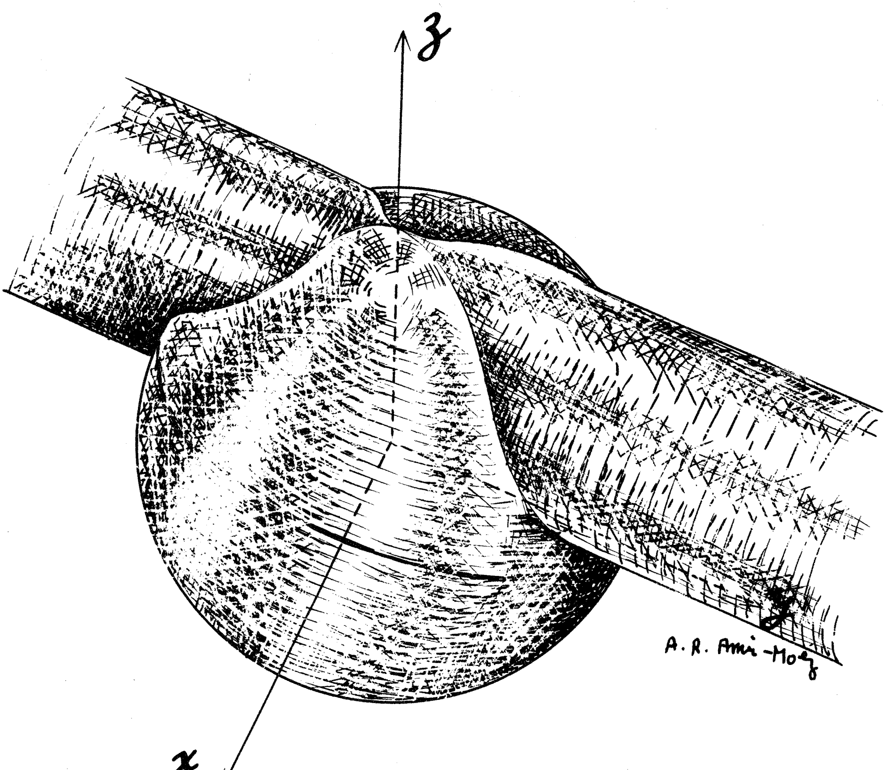
where $h=-\frac{1}{4}$, $g=-\frac{1}{4}\cos 3\phi$.

By factoring the left member of (4), we find the roots $\cos\phi$, $(-\cos\phi+\sqrt{3}\sin\phi)/2$, and $(-\cos\phi-\sqrt{3}\sin\phi)/2$. Now in order to solve (1), we use (3) and get $\cos^2 3\phi=-G^2/4H^3$, $\cos 3\phi=G/2H\sqrt{-H}$. From the theorem, $y_1=Ghy_2/gH$ or $y_1=2\sqrt{-H}\,y_2$. Now let $y_2$ take the values of the roots of (4) to obtain the roots of (1):

$$2\sqrt{-H}\cos\phi\,,\quad \sqrt{-H}(-\cos\phi+\sqrt{3}\sin\phi),\quad \sqrt{-H}\,(-\cos\phi-\sqrt{3}\sin\phi)$$

$$\text{where}\quad \phi=(1/3)\cos^{-1}(G/2H\sqrt{-H})\,.$$

The student will find that using the hyperbolic functions $\sinh\phi$ and $\sinh^{-1}\phi=\log_e(\phi+\sqrt{\phi^2+1})$ with the method employed above, will lead to Cardan's solution of (1) in the reducible case.




$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + z^2 = 2z \end{cases}$$

# COMMENTS ON PAPERS AND BOOKS

Edited by CHARLES W. TRIGG, Los Angeles City College

This department will present comments on papers published in the MATHMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to *Charles W. Trigg, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.*

## COMMENT ON MAXEY BROOKE'S "DIGITAL ROOTS OF PERFECT NUMBERS"*

B. L. SCHWARTZ, Monterey, California

The interesting result presented by Maxey Brooke in the article cited is rendered less forceful by several minor errors in the body of the development. These serve to make it difficult for any serious reader attempting to follow the argument, since, although the author uses numerous examples, he leaves all proofs to the reader. To assist those who desire to follow the development but are unfamiliar with the theory, it may be worthwhile to formulate correct statements for some of the places where the author has been inexact.

On page 100, two laws obeyed by digital roots are stated. They should be,

"1. The digital root of the sum of two numbers equals the digital root of the sum of the digital roots of the numbers.

"2. The digital root of the product of two numbers equals the digital root of the product of the digital roots of the numbers."

Near the bottom of the same page, the statement appears,

"The formula for a perfect number must be $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime."

This should be amended to read, "The formula for an *even* perfect number must be ···." This result is not elementary. The perfect character of all numbers of the form given is easily proved, but the converse, that all even perfect numbers are of that form, lies rather deeper. It was first proved by Euler. A simplified proof, due to Dickson, is indicated in Rouse Ball's *Mathematical Recreations and Essays*, on page 67 of the 1947 Macmillan edition. As far as I can see, this is the only result in the paper not easily proved by a persevering reader. The question of the existence of odd perfect numbers is at present unsolved.

The result stated at the end of the paper is not obviated by any of the foregoing, but there also a new restriction on $n$ must be introduced. The conclusion should read,

"Hence, for $n > 2$, $n$ must be odd before $2^{n-1}(2^n - 1)$ can be perfect. And the digital root of such a perfect number must be 1."

---

**Mathematics Magazine*, Vol. 34, No. 2, Nov.-Dec., 1960.

The first part of this conclusion is not new, and it does not require the use of digital roots for its proof. We need merely show, as Brooke does, that when $n$ is even, $2^n - 1$ is divisible by 3 and hence cannot be prime for $n > 2$. Let $n = 2k$. Then

$$2^{2k} - 1 = (2^k - 1)(2^k + 1) .$$

Now of the three consecutive integers $2^k - 1$, $2^k$, $2^k + 1$, exactly one is divisible by 3; but it clearly is not $2^k$; and the result follows.

The other part of Brooke's conclusion is particularly intriguing when we recall that the digital root is associated with the notational system. Brooke has assumed throughout a decimal system and is using a decimal digital root, i. e., the residue modulo 9. Other number systems, e. g., octal, duodecimal, binary, generate a different digital root function, but the property of perfectness of numbers is independent of the notation system. Brooke therefore has a hypothesis which is purely number theoretic and a conclusion which depends on the base of his number system. This appears to imply something special about the decimal base, for the property stated is easily shown not to be invariant with change of base. This suggests an interesting area for further study.

---

# A NOTE ON A COMMENT TO PROBLEMS AND QUESTIONS

WILLIAM E. CHRISTILLES,* St. Mary's University, San Antonio, Texas

In the 1961 March-April issue of the *Mathematics Magazine*, the author made a comment concerning the solution to problem 395, from the 1960 May-June issue of *Mathematics Magazine*, which stated that

$$f(n) = n^2 - n + 41 \tag{1}$$

is the product of at most two prime factors for $n \leqq 420$.

Note: Although certain quadratics assume prime values over a large range of values, no polynomial $f(n)$ with integral coefficients, not a constant, can be prime for all $n$. [1]

In that comment, several terms such as "division closure" and "closure for multiplication" were used without clarification. Also the author implied that his results were simple and did not involve tedious calculations which was not technically correct. Hence, in this short note the author will attempt to revise and clarify his previous comments.

From Dickson's Table 1 [2] and Theorem 70 [3], we know that every factor of a number of the form

$$x^2 + xy + 41y^2 , \tag{2}$$

of discriminant $d = -163$, in integers $x$ and $y$ with integral coefficients, is also of form (2). But the set of integers defined by form (1) is a subset of the set of integers which have representations in form (2). Hence, every factor of a number of form (1) has a representation by form (2). Now, considering all possible products of three or more factors, $p_j$, $p_u$, and $c_w$, none of which is unity, $p_j$ and $p_u$ are primes, and $c_w$ is a prime or composite, with

$$p_j p_u c_w \leqq (420)^2 - (420) + 41 , \tag{3}$$

we must show that $p_j p_u c_w \neq n^2 - n + 41$, for any $n \leqq 420$.

Now the smallest prime factor that

$$f(n) = f(420) = A \tag{4}$$

can have is 41. Then $A/41$ could be factorable into the product of at least two primes of form (2), the smallest of which cannot be smaller than 41. Hence, the largest factor of $A$ cannot exceed $A/41^2$, or 103.

Then one must consider the products of all primes of form (2) from 41 to 103, the primes taken three at a time and not necessarily distinct; and, either (a), generate $f(n)$ for the successive $n$'s and compare with the above products, or (b), attempt to put each product into the form (1).

If method (a) is used, we may observe that a number $N$ of form (2) can have three prime factors only if $N \geqq (41)^3$, since 41 is the smallest prime factor possible. Then one need only consider those $f(n) > (41)^3$,

*The author is indebted to the referee for several helpful suggestions.

that is, those $f(n)$ obtained from $n > 262$. We do not need to consider the equality since $(41)^3 = 68,921 = 68,880 + 41$, and 68,880 is not the product of two consecutive integers; hence, $(41)^3$ does not have a representation in form (1).

If method (b) is used, one can observe that $f(n) - 41 = n^2 - n$, which is the product of two consecutive integers, one being less than, the other being greater than $\sqrt{f(n) - 41}$. This fact can be used to test the products.

The author has tested this problem with the use of a Marchant fully-automatic desk calculator, employing method (a), thus verifying that "$f(n)$ contains at most two prime factors for $n \leqq 420$."

It is perhaps worthy of mention that since

$$(5)\quad (x^2 + bxy + cy^2)(u^2 + buv + cv^2) = (xu + bvx + cvy)^2 + b(xu + bvx + cvy)(uy - vx) + c(uy - vx)^2$$

$(41)^3$ must have a representation in form (2), although it has no representation in (1). To be more specific

$$(6)\quad (41)^3 = [(1)^2 + (1)(-1) + 41(-1)^2]^3 = (41)^2 + (41)(-41) + 41(-41)^2 ,$$

from (5).

*FOOTNOTES*

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* London, 1954, p. 18.
2. Leonard Eugene Dickson, *Introduction to the Theory of Numbers,* New York, 1929, p. 85.
3. *Ibid.,* p. 95.

## BOOK REVIEWS

*An Introduction to Linear Statistical Models,* Volume 1. By Franklin Graybill. McGraw-Hill Book Co., Inc., New York, 1961, 463 pages.

In recent years many fine books have appeared treating the area of experimental statistics. This is another such book.

The first four chapters review the basic tools needed in reading the remainder of the book. Chapter 1 deals essentially with Matrix Algebra which can be used with great profit as a review of certain parts of that subject. Chapter 2 covers the basic concepts of estimation and testing. Usually given in a first course in mathematical statistics, Chapter 3 discusses briefly the Multivariate Normal Distribution. Chapter 4 is concerned with the distribution of Quadratic Forms. This latter chapter is one of the few places in the literature where one can find any discussion of the

Noncentral Chi-square and $F$ distributions.

The main body of the book starts with Chapter 5 where the 5 linear models to be discussed are introduced. Chapters 6-8 cover Model 1 which is $Y = X\beta + e$ where $Y$ is a random observed vector, $X$ is an $n \times p$ matrix of known fixed quantities, $\beta$ is a vector of unknown parameters and $e$ is a random vector. Excluded from this general classification are the Experimental Design Models (these are the special cases when the matrix $X$ consists of just 0's and 1's) which are covered separately as Model 4 later on in the book. Chapter 9 deals with what the author calls the Functional Relationship model, that is the mathematical quantities $X$ and $Y$ are unobservable variables and are related by $Y = \alpha + \beta X$ ($\alpha$ and $\beta$ being unknown). What is observed are the random variables $x$, $y$ which are given by $y = Y + e$, $x = X + d$ where $e$ and $d$ are random errors. Chapter 10 covers Model 3 which is nothing more than Multiple Regression. Chapters 11-15 take up some of the experimental designs such as the Factorial Models and Incomplete Black Models. In Chapter 15 some discussion is made about the underlying assumptions of the above models and their necessity for the procedures used. Also covered in this chapter is Tukey's test for additivity. In Chapters 16 and 17, Model 5 which is usually known as Variance Components is treated. Chapter 18 gives an introduction to models which are mixtures of Models 1 and 4 and 4 and 5. The appendix has tables of the Central Chi-square and $F$ distribution, the student's $t$ and the Noncentral Beta distribution.

As a text for an experimental statistics course, this book would be a most excellent choice. It is well written and has that most important property that it is remarkably free from errors, both typographical and technical. The examples are well chosen and the problems are in general good choices.

The usage of this volume and the projected volume 2 as a reference work might be limited by the absence of such topics as "Missing Value and Related Formula", a thorough discussion of what is known about the effect of departures from the usual assumptions and "Nonparametric Techniques" in design of experiments.

In closing, it is felt that this book gives an excellent introduction to the area of experimental statistics and is well worth having.

---

J. E. Nylander, The Boeing Company

*Fundamentals of Mathematics*. By E. P. Vance. Addison-Wesley Publishing Company, Reading, Massachusetts, 1960, x+469 pp., $7.50.

*Fundamentals of Mathematics* is an excellent text for the college freshman whose high school mathematics training consists of first year algebra and plane geometry. It presents in a logically unified manner algebra, trigonometry, analytic geometry, and an introduction to differential and integral calculus.

Professor Vance has made no startling innovations, but the book has a freshness that makes it interesting to read. Proofs are presented with as much rigor as is suitable for a text at the elementary level. The exposition is clear, and the illustrations are well-chosen.

The unification of trigonometry and analytic geometry is based on the definition of the circular functions. Proper emphasis is laid on the analytic aspect of trigonometry, although the computational side is adequately covered. A very good chapter on determinants includes the definition of an $n$th order determinant and its expansion by minors. The notion of the derivative is developed fairly early and with great care. The importance of the derivative is stressed by its use in various types of problems. The single chapter devoted to integration introduces both the indefinite and the definite integral and includes the application of integration to problems involving area, volume, and work done by a variable force. The last chapter is a short one dealing with permutations, combinations, and probability. It is doubtful that enough explanation has been included to prepare the student for some of the exercises presented in this chapter.

Each section of the book contains a set of problems varying in difficulty. Many of them are routine, but some are rather challenging. Interesting historical material is incorporated in the body of the text.

No mention of the differential is made, though the symbol $dx$ is used in defining an integral. The symbol $Dy$ for the derivative seems to the reviewer less satisfactory in an elementary text than $D_x y$ or $dy/dx$. An occasional disagreement between a pronoun and its antecedent or between a verb and its subject detracts from an otherwise admirable book.

---

Janet McDonald, Vassar College

*Classical Mathematics*. By Joseph E. Hofmann. Philosophical Library, New York, 1959, 154 pp., $4.75. Translated from the German by H. O. Midonick.

Subtitled *A Concise History of the Classical Era in Mathematics*, this book is too small to be little more than a long list of names and dates, of investigations and discoveries, and of correspondance, publications and controversies of seventeenth and eighteenth century mathematicians. Into the limited space available, the author has packed the results of an apparently enormous amount of research, mentioning the works of about 375 mathematicians along with brief biographies of many of them. Fortunately, there are similar books available which, though less comprehensive, are more informative, more interesting, and more enjoyable.

---

Joseph M. C. Hamilton, Los Angeles City College

*Boundary Problems in Differential Equations*. Edited by Rudolph E. Langer. The University of Wisconsin Press, 1960, x + 324 pp., $4.00.

This book consists of nineteen papers written in this field, by experts from many parts of the world. To point out the actual value of the book, it would be proper to mention the authors, who are Richard Bellman, Garrett Birkhoff, Hans F. Bueckner, Lothar Collatz, Richard Courant, J. B. Diaz, Jim Douglas, Jr., Louis Ehrlich, Gaetano Fichera, Leslie Fox, K. O. Friedrichs, Paul R. Garabedian, Peter Henrici, W. T. Koiter, Johann Schröder, Ian N. Sneddon, Richard S. Varga, Calvin H. Wilcox and David Young.

These papers consist of the theoretical aspects of the subject and also the numerical methods. Several of these papers contain methods of unifying the hyperbolic, parabolic and elliptic cases. Numerical methods are presented in several papers.

A true review of this book takes many pages. But this book is a must for any library and for this price it is a great bargain.

---

Ali R. Amir-Moez, University of Florida

*The Contest Problem Book.* By Charles T. Salkind. Random House, New York, 1961, 154 pp., paper back, $1.95.

This is volume 5 of the New Mathematical Library published in cooperation with Yale University for the Monograph Project of the School Mathematics Study Group. Reproduced are the complete sets of problems proposed by the Mathematical Association of America in its annual contests from 1950 through 1960 for high school students. Answer keys for each examination are given, followed by solutions of each of the problems.

In order to facilitate the use of the book, a classification of the problems is provided according to the fields in the standard high school curriculum upon which the contests are based. This classification acts as an effective index. Location of particular problems is aided by clear page headings. The page arrangement and choice of type make for easy reading.

Interest in developing and administering the high school contests is based on "the firm belief that one way of learning mathematics is through selective problem solving." This volume should do much to encourage capable students in this type of learning when made available to them in school libraries or as presentation copies from those interested in their progress.

---

Charles W. Trigg, Los Angeles City College

## BOOKS RECEIVED FOR REVIEW

*Apollonius of Perga.* By T. L. Heath. Barnes and Noble, Inc., New York, 1961, clxx + 254 pages. $9.00.

*Numbers: Rational and Irrational.* By Ivan Niven. Random House, New York, 1961, viii + 136 pages. $1.95.

*What is Calculus About?* By W. W. Sawyer. Random House, New York, 1961, 118 pages. $1.95.

*An Introduction to Inequalities.* By Edwin Beckenbach and Richard Bellman. Random House, New York, 1961, 133 pages. $1.95.

*Geometric Inequalities.* By Nicholas D. Kazarinoff. Random House, New York, 1961, 132 pages. $1.95.

*The Lore of Large Numbers.* By Philip J. Davis. Random House, New York, 1961, x+169 pages. $1.95.

*Science Awakening.* By B. L. Van der Waerden. Translated by Arnold Dresden. Oxford University Press, New York, 1961. $7.50.

*More Numbers : Fun and Facts.* By J. Newton Friend. Charles Scribner's Sons, New York, 1961. $2.95.

*Tables of Weber Functions.* By I. Y. Kireyeva and K. A. Karpov. Pergamon Press, New York, 1961, xiv+364 pages. $20.00.

*Tables of Spectrum Lines.* By A. N. Zaidel, V. K. Prokof'ev, and S. M. Raiskii. Pergamon Press, New York, 1961, xliii+554 pages. $14.00.

*Stability in Non Linear Control Systems.* By A. M. Letov. Translated by J. George Adashko. Princeton University Press, Princeton, 1961, 316 pages. $8.50.

*Theory of Elasticity.* By V. V. Novozjilov. Translated by J. K. Lusher. Pergamon Press, New York, 1961, xii+460 pages. $12.50.

# AN IDEMPOTENT MATRIX

J. L. STEARN, U. S. Coast and Geodetic Survey

An interesting matrix arises in the development of certain covariance matrices where the mathematical model is constructed by a system of conditions with parameters.

Let

$$\underset{rs}{A}\ \underset{s1}{v} + \underset{rp}{B}\ \underset{p1}{u} + \underset{r1}{t} = 0 \qquad r > p,\ s > r-p$$

be a least squares solution minimizing the function

$$F = v'Wv - 2(Av + Bu + t)'k$$

giving

$$k = -(AW^{-1}A')^{-1}(Bu+t)\ .$$

A prime as a superscript denotes transpose and a negative unit as a superscript denotes inverse. In the model, $v$ is a vector of residuals, $u$ is a vector of unknown parameters and $t$ is a vector of absolute terms. The vector $k$ introduced in the least squares solution is the Lagrangian vector of unknowns. $W$ is a matrix of weights assigned to the residuals a priori.

It is known that for a system of conditions $Av + t = 0$, the covariance matrix of the vector $k$ written $\Sigma_{kk'}$ is given by

$$\sum_{kk'} = \sigma_0^2(AW^{-1}A')^{-1}\ ,$$

where $\sigma_0^2$ is the unit variance. For the model $Av + Bu + t = 0$ we find that

$$\sum_{kk'} = \sigma_0^2(AW^{-1}A')^{-1} \cdot H$$

where

$$H = I - BR^{-1}B'(AW^{-1}A')^{-1} \quad \text{and} \quad R = B'(AW^{-1}A')^{-1}B\ ;$$

$I$ is the identity matrix.

The matrix $H$ is idempotent. That is

$$H = H \cdot H = H^n\ .$$

*Proof:*

$$(I - BR^{-1}B'(AW^{-1}A')^{-1})(I - BR^{-1}B'(AW^{-1}A')^{-1}) = I - BR^{-1}B'(AW^{-1}A')^{-1} - BR^{-1}B'(AW^{-1}A')^{-1} + BR^{-1}B'(AW^{-1}A')^{-1}BR^{-1}B'(AW^{-1}A')^{-1}$$

But

$$B'(AW^{-1}A')^{-1}B = R\ ,$$

hence the last term reduces to

$$BR^{-1}RR^{-1}B'(AW^{-1}A')^{-1} \quad \text{or} \quad BR^{-1}B'(AW^{-1}A')^{-1}$$

leaving $I-BR^{-1}B'(AW^{-1}A')^{-1}$; showing that $H$ is idempotent, i. e., $H = H \cdot H = H^n$.

$I-H$ is also idempotent; thus

$$(I-H)(I-H) = I-H-H+H \cdot H = I-H-H+H = I-H \ .$$

If one considers the model

$$2v_1 - v_2 + 3v_3 + v_4 + 4u_1 - u_2 + 1.8 = 0$$

$$v_1 + 3v_2 + v_3 - 2v_4 + u_1 + 6u_2 - 3.6 = 0$$

$$5v_1 + 2v_2 - v_3 + 8v_4 - 2u_1 - 3u_2 + 6.4 = 0$$

with

$$W = \begin{bmatrix} 1 & & & \\ & .3333 & & \\ & & .5 & \\ & & & .5 \end{bmatrix}$$

then it turns out that

$$H = \begin{bmatrix} .0422 & .0655 & .1165 \\ .0168 & .0270 & .0485 \\ .3354 & .5213 & .9309 \end{bmatrix} .$$

It is easily shown that $H$ reproduces itself and that the determinant value of $H$ is zero.

---

*(Continued from page 38.)*

are: 3, 13, 31, 37, 61, 67, 73, 151, 157, 193, 271, 283, 331, 367, 433, 487, 523, 577, 601, 613, 643, 661, 727, 751, and 823.

Of the fractions with denominators less than 100, the one most closely approximating $e$ by defect is $(106)/(39) \doteq 2.717949$, being accurate to three decimal places. The fraction most closely approximating $e$ by excess is $(193)/(71) \doteq 2.718310$, which is accurate to four decimal places.

---

# PROBLEMS AND SOLUTIONS

Edited by ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.*

**467.** *Proposed by C. W. Trigg, Los Angeles City College.*

Identify the unique four-digit non-palindromic strobogrammatic integer which is the sum of two positive cubes. (A strobogrammatic integer reads the same after rotation through 180°, e. g. 89068.)

**468.** *Proposed by Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

The three perimeter bisecting lines, each of which passes through a corresponding midpoint of a side of a given triangle, meet at a point $S$. The three perimeter bisecting lines, each of which passes through a corresponding vertex of the given triangle meet in a point $N$. Show that $S$ is the midpoint of the line segment joining the incenter of the given triangle to the point $N$.

**469.** *Proposed by J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela.*

A random straight line is drawn across a regular hexagon. What is the probability that it intersects two opposite sides?

**470.** *Proposed by William Squire, West Virginia University.*

Prove that

$$\sum_{n=0}^{N} (-1)^n \frac{(1-2N-n)!(1+N)!}{(1+n)!(N-n)!(1+N-n)!(2-2N)!} = \frac{1}{(N+2)!} .$$

**471.** *Proposed by Brother U. Alfred, St. Mary's University, California.*

Determine all the prime numbers between one and two million for which $[N/2]+[N/2^2]+[N/2^3]+\cdots = N-3$ where the square brackets represent "the largest integer in" and the dots indicate that the process is to be carried as far as possible.

**472.** *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Let $(C)$ be a conic and $M$ be a variable point on it. Let $T$ be the point symmetric to $M$ with respect to the main axis, and $t$ the tangent line at $T$. Denote the intersection of the perpendicular from $M$ to $t$ with the line joining $T$ to the center of the conic by $I$. If $M'$ is symmetric to $M$ with respect to $I$, prove that

1. The locus of $M'$ is another conic $(C')$ of the same kind as $(C)$.
2. The conics $(C)$ and $(C')$ are confocal.

**473.** *Proposed by Joseph W. Andrushkiw, Seton Hall University.*
Show that

$$\frac{\displaystyle\int_0^{2\pi} \frac{x^2 \sin x\, dx}{1+\sin^2 x}}{\displaystyle\int_0^{2\pi} \frac{x \sin x\, dx}{1+\sin^2 x}} = 2\pi .$$

## SOLUTIONS

### Late Solutions

**439.** *J. A. H. Hunter, Toronto, Ontario, Canada.*

### A Dedicated Cryptarithm

**446.** [May, 1961] *Proposed by David L. Silverman, Fort Meade, Maryland.*
Find the digital equivalents of the letters in the cryptaddition

```
 THREE
 EIGHT
  NINE
------
TWENTY .
```

(Dedicated to 6.0. 14522.)

*Solution by W. C. Waterhouse, Harvard University.*
Clearly $T = 1$, so $W = 0$. If $E = 9$, $Y = 9$; so $E = 8$, $Y = 7$. Then $H + N = 12$; so $H = 3$, $N = 9$ or $H = 9$, $N = 3$. It is then easy to find the solution

```
 19488
 85291
  3538
------
108317 ,
```

unique except for the possibility of interchanging $E$ and $G$. A more rapid solution is available to those who guess *a priori* that the dedication is to C. W. Trigg. This dedication also makes the above solution unique.

*Also solved by Merrill Barneby, University of North Dakota, Joseph B. Bohac, St. Louis, Missouri; D. A. Breault, Sylvania Electronics Systems, Waltham, Massachusetts; Maxey Brooke, Sweeney, Texas; J. L. Brown, Jr., Pennsylvania State University; Stephen M. Call, Harvard University; Daniel I. A. Cohen, Philadelphia, Pennsylvania; N. A. Court,* *University of Oklahoma; Monte Dernham, San Francisco, California; David Drasin, Temple University; H. W. Gould, West Virginia University; J. A. H. Hunter, Toronto, Ontario, Canada; Kirk Jensen, Gridley Union High School, California; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Sidney Kravitz, Dover, New Jersey; Gilbert Labelle, Longueuil's College, P. Q., Canada; Herbert R. Leifer, Pittsburgh, Pennsylvania; James W. Mellender, University of Wisconsin-Milwaukee; Francis L. Miksa, Aurora, Illinois; John W. Milson, Texas A and I College; C. C. Oursler, Southern Illinois University; C. F. Pinzka, University of Cincinnati; Lawrence A. Ringenberg, Eastern Illinois University; Chris B. Schaufele, Florida State University; J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela; Donald R. Simpson, University of Alaska; James L. Solomon, Jr., Morris College, South Carolina; William Squire, Southwest Research Institute, San Antonio, Texas; Alan Sutcliffe, Knottingley, Yorkshire, England; R. W. Swesnik, Dallas, Texas (Partially); P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.; C. W. Trigg, Los Angeles City College; Honer Webb, Bucknell University; Dale Woods, Northeast Missouri State Teachers College; and the proposer.*

Dernham pointed out that this is an example of a *charming* cryptarithm. See the *American Mathematical Monthly*, 1947, p. 413, problem E 751 and Editor's Note.

Hunter stated that this cryptarithm is "alphametic", a term that has been widely adopted for such doubly true cryptarithms.

## An Arbelos

**447.** [May, 1961] *Proposed by James W. Mellender, University of Wisconsin.*

Given two circles of radius $x$ and $y$ which are tangent externally and their circumcircle. Determine the radius of the circle tangent to the three given circles.

**I.** *Solution by C. N. Mills, Sioux Falls College, South Dakota.*

This problem is a special case of the famous Apollonius problem, given three circles externally tangent two by two, to determine the radii of the two circles tangent to the given circles. If the radii of the given circles are $a$, $b$, and $c$, then the radii of the required circles are given by the formula

$$R = \frac{abc}{2\sqrt{[abc(a+b+c)]} \pm (ab+ac+bc)}.$$

The minus sign gives the radius of the circumcircle. In this problem we have the circumcircle of the two given circles and the required circle of radius $R$. Hence

$$a+b = \frac{xyR}{2\sqrt{[xy(x+y+R)]} - (xy+xR+yR)}.$$

Simplifying the equation, after considerable algebraic manipulation, we get

$$R = \frac{xy(x+y)}{x^2+xy+y^2}.$$

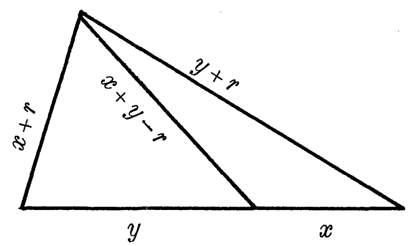
**II.** *Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

The given conditions lead to the triangle as shown where $r$ is the radius of the desired circle. Applying Stewart's theorem one obtains

$$(x+y-r)^2(x+y) = y(y+r)^2 + x(x+r)^2 - xy(x+y)$$

which when solved for $r$ gives

$$r = \frac{xy(x+y)}{x^2+y^2+xy}.$$




**III.** *Solution by C. F. Pinzka, University of Cincinnati.*

Let $r$ be the unknown radius and $A, B, C, D$ the centers of circles with radii $r, x, y, x+y$. Since triangles $ABD$ and $ACD$ have a common side and equal perimeters, the ratio of their areas simplifies to $\sqrt{x(y-r)/y(x-r)}$ by Heron's formula. But the ratio of the areas is also $y/x$, whence

$$r = \frac{xy(x+y)}{x^2+xy+y^2}.$$

*Also solved by Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeney, Texas; J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela; Lt. Donnelly J. Johnson, U. S. Air Force Academy; Joseph D. E. Konhauser, H. R. B.-Singer, Inc., State College, Pennsylvania; Robert P. Goldberg, Brooklyn, New York; Alan Sutcliffe, Knottingley, Yorkshire; and the proposer.*

## A Square Sum of Squares

**448.** [May, 1961] *Proposed by Brother U. Alfred, St. Mary's College, California.*

Determine an infinite series of terms such that each term of the series is a perfect square and the sum of the series at any point is a perfect square.

*Solution by Anthony F. Dugan, Lockheed Missiles and Space Company,* *Palo Alto, California.*

Let the series be denoted by:

$$S = a_1^2 + a_2^2 + a_3^2 + \cdots .$$

Also let

(1) $$(a_1, a_2, \cdots) = 1 .$$

If this is not so, the common factor can be taken out of each term. The following system can now be generated:

$$a_1^2 + a_2^2 = b_2^2$$

$$a_1^2 + a_2^2 + a_3^2 = b_3^2$$

$$b_2^2 + a_3^2 = b_3^2$$

$$\cdots$$

$$b_N^2 + a_{N+1}^2 = b_{N+1}^2 .$$

Now, assume

$$b_{N+1} = a_{N+1} + C .$$

Then

$$(a_{N+1} + C)^2 - a_{N+1}^2 = b_N^2 ,$$

$$a_{N+1} = \frac{b_N^2 - C^2}{2C} .$$

This can have no solution in integers for all $N$ unless $C \mid a_i$ for all $i$. Therefore, $C = 1$,

$$a_{N+1} = \frac{b_N^2 - 1}{2}$$

which is integral if $b_N$ is odd. The series is, therefore, generated by starting with any prime greater than 2. For example:

(2) $$S = 3^2 + 4^2 + (12)^2 + (84)^2 + \cdots .$$

If $a_1 = 5$, the series is not essentially different:

(3) $$S' = 5^2 + (12)^2 + (84)^2 + \cdots .$$

A unique series can be generated for any prime which is not a partial sum of the series (2). E.g.,

$$S = 7^2 + (24)^2 + (312)^2 + \cdots .$$

And the partial sum $S_N = (a_N + 1)^2$.

*Also solved by Donald K. Bissonnette, Florida State University; D. O. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; J. L. Brown, Jr., Pennsylvania State University; E. F. Canaday, Meredith College, North Carolina; L. Carlitz, Duke University; Daniel I. A.*

*Cohen, Philadelphia, Pennsylvania; D. Drashin, Temple University; J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela; John M. Howell, Los Angeles City College; Vladimir F. Ivanoff, San Carlos, California; Murray S. Klamkin, AVCO, Wilmington, Massachusetts; David A. Klarner, Humbolt State College, California; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Sidney Kravitz, Dover, New Jersey; Herbert R. Leifer, Pittsburgh, Pennsylvania; C. C. Oursler, Southern Illinois University; C. F. Pinzka, University of Cincinnati; L. A. Ringenberg, Eastern Illinois University; William Squire, West Virginia University; Alan Sutcliffe, Knottingley, Yorkshire, England; W. C. Waterhouse, Harvard University; and the proposer.*

## A Logarithmic Inequality

**449.** [May, 1961] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

For what positive integral $x$ and for what $y = y(x)$ is the following inequality true?

$$(1-\tfrac{1}{2}x)(2x)^{2-2y} \leqq 1 .$$

*Solution by Gilbert Labelle, Longueuil's College, P. Q., Canada.*

Suppose that all the integers $x > 0$ satisfy the inequality

$$(1-\tfrac{1}{2}x)(2x)^{2-2y} \leqq 1 ,$$

then we have

$$\frac{2x-1}{2x}\cdot(2x)^{2-2y} \leqq 1 \quad \text{or} \quad \frac{2x-1}{(2x)^{2y-1}} \leqq 1 ,$$

since $x$ is positive we have $(2x)^{2y-1} > 0$ for all $2y-1$ as exponent; then,

$$(2x-1) \leqq (2x)^{2y-1} ,$$

taking logarithms of both sides we have:

$$\log(2x-1) \leqq (2y-1)\log(2x) \quad \text{or} \quad 2y-1 \geqq \frac{\log(2x-1)}{\log 2x} .$$

Then

$$2y \geqq \frac{\log(2x-1)}{\log 2x} + 1 .$$

Thus

$$y \geqq \tfrac{1}{2}\left(1+\frac{\log(2x-1)}{\log 2x}\right) .$$

Since this quantity is always positive for all integers $x > 0$ we can put:

(1) $$y = \tfrac{1}{2}K\left(1+\frac{\log(2x-1)}{\log 2x}\right) \qquad (K \geqq 1) .$$

## Convergent Subseries

**451.** [May, 1961] *Proposed by B. L. Schwartz, Monterey, California.*

Let $S = \Sigma x_n$ be any conditionally convergent series of real terms. Let $r$ be any real number. Prove there exists a conditionally convergent subseries $S'$ of $S$ (obtained by deletion of terms without rearrangement) which converges to $r$.

*Solution by J. L. Brown, Jr., Pennsylvania State University.*

Proceed termwise deleting all negative terms until the sum of the retained positive terms first exceeds $r$. (This is possible since the series consisting of all positive terms of the series must diverge; the same is true for the series of negative terms.) From that point, delete positive terms until the sum of all retained terms first falls below $r$. Repeat this alternation of steps indefinitely. The difference at the end of each step between $r$ and the partial sum of the subseries at that point is less in magnitude than the magnitude of the last retained term. Since $x_n \to 0$ as $n \to \infty$, the partial sums converge to $r$.

*Also solved by Murray S. Klamkin, AVCO, Wilmington, Massachusetts; Lawrence A. Ringenberg, Eastern Illinois University; and the proposer.*

## A Matrix Group

**452.** [May, 1961] *Proposed by H. Schwerdtfeger, McGill University, Montreal.*

Prove that all regular $n$ by $n$ matrices $A$ with complex elements such that a certain complex vector $x$ is eigen vector of $A$ ($Ax = \alpha x$, with complex eigen value $\alpha$) form a group $\mathbf{G}_x$ with respect to matrix multiplication.

*Solution by W. C. Waterhouse, Harvard University.*

Let $Ax = \alpha x$; then

$$x = A^{-1}Ax = A^{-1}\alpha x = \alpha A^{-1}x\ ,$$

so $A^{-1}x = \alpha^{-1}x$ (since $A$ is regular, of course $\alpha \neq 0$) and $A^{-1} \epsilon\ \mathbf{G}_x$. If $Bx = \beta x$,

$$ABx = A\beta x = \alpha\beta x\ ,$$

so $AB \epsilon\ \mathbf{G}_x$. Thus $\mathbf{G}_x$ is a subgroup of the multiplicative group of regular matrices.

*Also solved by L. Carlitz, Duke University; F. D. Parker, University of Alaska; and the proposer.* The proposer pointed out that for

$$x = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

the theorem provides an easy solution to Problem 384, this magazine, September 1959, p. 51 and March 1960, p. 230. Regarding Problem 384 he also refers to the note by A. Wilansky, "The Row Sums of the Inverse Matrix", *American Mathematical Monthly*, Vol. 58, 1951, p. 614.

## Comment on Problem 423.

**423.** [September 1960 and September 1961] The proposer states that the point of the problem may be lost if it is not pointed out that the two inequalities are, in fact, each unified versions of the three triangular inequalities $a < b+c$, $b < a+c$, and $c < a+b$.

## Comment on Problem 437

**437.** [January 1961 and September 1961] H. W. Gould points out that Problem 437 is essentially the same as problem E 706 in the *American Mathematical Monthly*. It is also equivalent to the results found by J. W. L. Glaisher, "On the residue of a binominal-theorem coefficient with respect to a prime modulus," *Quarterly Journal of Math.,* 30 (1899), pp. 150-156. And it appears as problem 12, page 20, Chapter I, I. M. Vinogradov, *Elements of Number Theory*, Dover Publ., 1954. Other references could be supplied.

It may be of interest to rewrite it as follows :

$$\sum_{k=0}^{n} (-1)^{\binom{n}{k}} = n+1-2^{1+N(n)} ,$$

where $N(n)$ = the number of 1's in the binary expansion of $n$. I prefer to avoid the bracket notation which is apt to be confounded with the greatest integer notation.

It is also easy to see that

$$\sum_{k=0}^{n-1} (-1)^{\binom{2n}{2k+1}} = n , \quad \text{for } n \geqq 1 .$$

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 290.** Through a given point within a given angle, construct a line which will form a triangle of minimum area. [*Submitted by Murray S. Klamkin.*]

**Q 291.** What number, when divided into 1108, 1453, 1844, and 2281, always leaves the same remainder? [*Submitted by C. W. Trigg.*]

**Q 292.** Prove that cos 1° is irrational. [*Submitted by Leo Moser.*]

**Q 293.** Is the number 12345679 composite? [*Submitted by Murray S. Klamkin.*]

## Erratum

In **Q 289** on page 435, Vol. 34, No. 7, November, 1961, in the second line the equation $BC = AC$ should read $BD = AC$.

## TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

**T 49.** Complete the Ideo-mathematical proportion $\Sigma : \Pi = {}_nC_2 : ?$ where $\Sigma$ means summation of a given set of quantities, $\Pi$ their product, and ${}_nC_2$ is the number of combinations of $n$ things taken two at a time. [*Submitted by Brother Alfred.*]

**T 50.** The two equations

$$x^{15} + ax + b = 0 \quad (a \text{ and } b \text{ integers})$$

and

$$x^{13} - 223x - 144 = 0$$

have a common factor. Find it. [*Submitted by Murray S. Klamkin.*]

**T 51.** Complete the Ideo-mathematical proportion

$$\Sigma : \Pi = ? : \frac{(2n)!}{2^n n!}.$$

[*Submitted by Brother Alfred.*]

*(Answers to Quickies and Solutions for Trickies are on page 62.)*

---

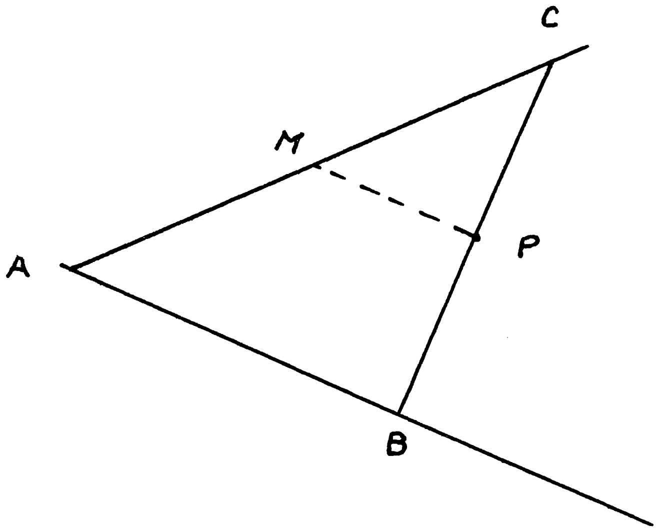
## ERRATUM

In the solution to problem 440 in Vol. 34, No. 7, page 427, the radicand in the last radical on that page should read $n^2 - (3/4)p^2$.

---

## ANSWERS *to Quickies on page 60*

**A 290.** In order for triangle $ABC$ to be a minimum it follows that $BP = PC$. Consequently, draw $PM$ parallel to $AB$, lay off $MC = MA$, and draw $CPB$.




**A 291.** $1453 - 1108 = 345$, $1844 - 1453 = 391$, $2281 - 1844 = 437$. Then $437 - 391 = 391 - 345 = 46 = 2(23)$. So 23 is the required divisor, since

$$(N_1 d + r) - (N_2 d + r) = d(N_1 - N_2) .$$

The remainder is 4.

**A 292.** Using DeMoivre's theorem, $\cos 30°$ can be expressed as a polynomial with integral coefficients in $\cos 1°$. Hence, $\cos 1°$ rational would imply that $\cos 30° = \sqrt{3}/2$ is rational.

## SOLUTIONS *for Trickies on page 61*

**S 49.** ${}_nC_2$ is the sum of the first $n-1$ integers. Therefore the fourth proportional is the product of these same integers or $(n-1)!$.

**S 50.** The usual method through successive reduction of degree of the equation would be too messy. However, if one notes that 144 and 233 are the 12th and 13th terms of the Fibonacci sequence 1, 1, 2, 3, 5, ···, it follows that $x^2 - x - 1$ is a factor of $x^{13} - 233x - 144$. This follows from $x^n = F_n x + F_{n-1}$ if $x^2 = x + 1$. Consequently, $x^2 - x - 1$ can also be a factor of $x^{15} + ax + b$. It is possible that there may be another common factor for the proper choice of $a$ and $b$ but it seems doubtful.

**S 51.** $(2n)!/2^n n!$ is the product of the first $n$ odd integers so the third member of the proportion is the sum of the first $n$ odd integers or $n^2$.

**Q 293.** Is the number 12345679 composite? [*Submitted by Murray S. Klamkin.*]

## Erratum

In **Q 289** on page 435, Vol. 34, No. 7, November, 1961, in the second line the equation $BC = AC$ should read $BD = AC$.

## TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

**T 49.** Complete the Ideo-mathematical proportion $\Sigma : \Pi = {}_nC_2 : ?$ where $\Sigma$ means summation of a given set of quantities, $\Pi$ their product, and ${}_nC_2$ is the number of combinations of $n$ things taken two at a time. [*Submitted by Brother Alfred.*]

**T 50.** The two equations

$$x^{15} + ax + b = 0 \quad (a \text{ and } b \text{ integers})$$

and

$$x^{13} - 223x - 144 = 0$$

have a common factor. Find it. [*Submitted by Murray S. Klamkin.*]

**T 51.** Complete the Ideo-mathematical proportion

$$\Sigma : \Pi = ? : \frac{(2n)!}{2^n n!}.$$

[*Submitted by Brother Alfred.*]

*(Answers to Quickies and Solutions for Trickies are on page 62.)*

---

## ERRATUM

In the solution to problem 440 in Vol. 34, No. 7, page 427, the radicand in the last radical on that page should read $n^2 - (3/4)p^2$.

---

## PROFESSIONAL OPPORTUNITIES

## IN MATHEMATICS

*Fifth Edition* *September 1961*

A completely revised version of an article which appeared originally in the AMERICAN MATHEMATICAL MONTHLY.

32 pages, paper covers

25¢ for single copies; 20¢ each for orders of five or more.

*Send orders to:*

HARRY M. GEHMAN, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

# THE MATHEMATICAL ASSOCIATION OF AMERICA

The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership of over 11,000, including more than 500 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the *American Mathematical Monthly*. Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs and for subscriptions to several journals.

Further information about the Association, its publications and its activities may be obtained by writing to:

HARRY M. GEHMAN, *Executive Director*
Mathematical Association of America
*University of Buffalo*
*Buffalo 14, New York*